Analysis for Dissipative Maxwell-Bloch Type Models

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Abstract

This thesis deals with the mathematical modeling of semi-classical matter-light interaction. In the semi-classical picture, matter is described by a density matrix ρ, a quantum mechanical concept. Light on the other hand, is described by a classical electromagnetic field $(\mathbf{E}, \mathbf{H})$. The evolution of density matrices is governed by the Liouville-von-Neumann equation, or versions of this equation. The evolution of an electromagnetic field is governed by the time dependent Maxwell equations.

We introduce a mathematical framework in which we state a systematic approach to include dissipative effects in the Liouville-von-Neumann equation. This framework is related to the GENERIC framework (General Equations for Non-Equilibrium Reversible Irreversible Coupling), which is mainly due to A. Mielke and H. C. Öttinger. The striking advantage of our approach is the intrinsic existence of a Liapunov function for solutions to the resulting evolution equation. Next, we couple the resulting equation to the Maxwell equations and arrive at a new self-consistent dissipative Maxwell-Bloch type model for semi-classical matter-light interaction. In the same mathematical framework we formulate a reduced Maxwell-Bloch type model.

The main focus of this work lies on the intensive mathematical study of both the dissipative and the reduced Maxwell-Bloch type model. Since both our models lack Lipschitz continuity, we create regularized versions of the models that are Lipschitz continuous. We mostly restrict our analysis to these Lipschitz continuous regularizations.

For regularized versions of the reduced model, we prove an existence theorem for solutions to the corresponding Cauchy problem with different types of initial conditions. For a restrictive set of initial conditions, we base our analysis on the Banach fixed point theorem. In this case we can also show that solutions are unique. For more general initial conditions, the usage of the Banach fixed point theorem is not possible. In this case, our analysis is based on a version of the div-curl lemma which is due to S. Conti, G. Dolzmann and S. Müller. Moreover, assuming that solutions exist, we perform a study of the long-time behavior of solutions to the original (not regularized) Cauchy problem. Here, our analysis starts from the Liapunov function for the solutions.

For regularized versions of the dissipative Maxwell-Bloch type model, we prove existence of solutions to the corresponding Cauchy problem. As in the latter case of the reduced model, the usage of the Banach fixed point theorem is not possible. This time, the core of the proof is based on results from compensated compactness due to P. Gérard and a Rellich type lemma. In parts, this proof closely follows the lines of an earlier work due to J.-L. Joly, G. Métivier and J. Rauch.

Zusammenfassung

Die vorliegende Dissertation befasst sich mit der mathematischen Modellierung semiklassischer Licht-Materie-Interaktion. Im semiklassischen Bild wird Materie durch eine Dichtematrix ρ beschrieben. Das Konzept der Dichtematrizen ist quantenmechanischer Natur. Auf der anderen Seite wird Licht durch ein klassisches elektromagnetisches Feld $(\mathbf{E}, \mathbf{H})$ beschrieben. Die zeitliche Evolution von Dichtematrizen wird durch die Liouville-von-Neumann-Gleichung bzw. durch Versionen dieser Gleichung beschrieben. Die Evolution elektromagnetischer Felder wird durch die zeitabhängigen Maxwell-Gleichungen beschrieben.

Wir stellen einen mathematischen Rahmen vor, in dem wir systematisch dissipative Effekte in die Liouville-von-Neumann-Gleichung inkludieren. Dieser mathematische Rahmen ist eng mit dem GENERIC-Konzept verwandt, welches im Wesentlichen durch A. Mielke und H. C. Öttinger eingeführt wurde. Bei unserem Ansatz sticht ins Auge, dass Lösungen der resultierenden Gleichung eine intrinsische Liapunov-Funktion besitzen. Anschließend koppeln wir die resultierende Gleichung mit den Maxwell-Gleichungen und erhalten ein neues selbstkonsistentes, dissipatives Modell vom Maxwell-Bloch-Typ. Im gleichen mathematischen Rahmen formulieren wir auch ein reduziertes Modell vom Maxwell-Bloch-Typ.

Der Fokus dieser Arbeit liegt auf der intensiven mathematischen Studie des dissipativen und des reduzierten Modells vom Maxwell-Bloch-Typ. Da beide Modelle Lipschitz-Stetigkeit vermissen lassen, kreieren wir regularisierte Versionen der Modelle, die Lipschitz-stetig sind. Wir beschränken unsere Analyse im Wesentlichen auf die Lipschitz-stetigen Regularisierungen.

Für die regularisierten Versionen des reduzierten Modells zeigen wir Existenzsätze für Anfangswertprobleme mit verschiedenen Typen von Anfangsdaten. Für eine etwas restriktivere Menge von Anfangsdaten basiert unsere Analyse auf dem Banach'schen Fixpunktsatz. Für allgemeinere Anfangsdaten lässt sich der Banach'sche Fixpunktsatz nicht anwenden. In diesem Fall basiert unsere Analyse auf einer Version des Div-Curl-Lemmas von S. Conti, G. Dolzmann und S. Müller. Unter der Annahme, dass Lösungen existieren, untersuchen wir das Langzeitverhalten des ursprünglichen (nicht regularisierten) Anfangswertproblems. Hier basiert unsere Analyse auf der Tatsache, dass Lösungen eine Liapunov-Funktion besitzen.

Für regularisierte Versionen des dissipativen Modells zeigen wir die Existenz von Lösungen des zugehörigen Anfangswertproblems. Wie im zweiten Fall des reduzierten Modells, ist die Anwendung des Banach'schen Fixpunktsatzes nicht möglich. Diese Mal besteht der Kern des Existenzbeweises aus einem Resultat von "compensated compactness", das von P. Gérard bewiesen wurde, sowie aus einem Lemma vom Rellich-Typ. In Teilen folgt dieser Beweis dem Vorgehen einer älteren Arbeit von J.-L. Joly, G. Métivier und J. Rauch.

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Contents

Introduction

This work deals with semi-classical modeling of matter-light interaction. Matter-light interaction plays a major role in the functionality of a laser. By semi-classical we mean that on the one hand, matter is described by a density matrix ρ, a quantum mechanical concept. Light on the other hand, is described by a classical electromagnetic field $\big(\mathbf{E}, \mathbf{H}\big)$.

In a laser an optically active material admitting N energy levels interacts with light. The energy levels of the active material are quantum states. In principle, it is not necessary to restrict the considerations to finite N. However, in the case where N is finite, one describes the active material with a density matrix ρ, a complex positive semi-definite Hermitian $N \times N$-matrix with $\mathrm{Tr}\rho = 1$. The evolution of density matrices is governed by the Liouville-von-Neumann equation, an ordinary differential equation. When coupled to a space dependent system, the Liouville-von-Neumann equation is understood as an equation depending parametrically on the space variable x.

In semi-classical modeling of matter-light interaction, the Liouville-von-Neumann equation is coupled to the time dependent Maxwell equations that govern the evolution of the electromagnetic field $\big(\mathbf{E}, \mathbf{H}\big)$ describing light. The Maxwell equations form a symmetric hyperbolic system of first order partial differential equations. The resulting coupled system is known as the Maxwell-Bloch system.

In the literature several versions of simplified models are studied. In particular, in the two-level case one introduces alternative variables for the density matrix and eliminates some of them. Also the Maxwell equations are often simplified. Then, one usually ends up with an inhomogeneous wave equation for the electric field $\mathbf{E}$, only.

The highlights and main novelties of this work are the systematic construction and the mathematical study of a dissipative Maxwell-Bloch type model as well as the mathematical study of a reduced Maxwell-Bloch type model. The thesis is organized as follows. In Chapter 1 we give a short introduction to semi-classical modeling of matter-light interaction, we derive some simplified models and give a short overview of the key (mathematical) literature on this topic. In Chapter 2 we introduce the mathematical framework of a new approach to modeling matter-light interaction. We derive a Maxwell-Bloch type model that couples a dissipative version of the Liouville-von-Neumann equation to the Maxwell equations. Furthermore, we propose a reduced Maxwell-Bloch type model that exhibits some structural similarities to the former model and fits in our mathematical framework. In Chapter 3 we give an analytical study of the reduced Maxwell-Bloch type model including an analysis of its long time behavior. In Chapter 4 we establish an existence result of a regularized version of the dissipative Maxwell-Bloch type model.

1. Semi-classical Modeling of Matter-Light Interaction

This chapter contains a short introduction to the semi-classical modeling of matter-light interaction. We introduce the prevailing coupling mechanism, present a common simplification and derive a reduced model. In Section 1.1 we give a short presentation of the concept of density matrices, state their properties and how their evolution is governed. We highlight the simple structure of the two-level case and discuss the effect of incident electric fields. In Section 1.2 we derive the classical Maxwell-Bloch system, which has been intensively studied in the literature. In Section 1.3 we give a brief and selective insight into some relevant literature. Finally, in Section 1.4 we derive a reduced version of the classical Maxwell-Bloch equations, which motivates a new non-linear model that will be studied in Chapter 3.

1.1. Evolution of Statistical Mixtures of Quantum States and Density Matrices

Statistical mixtures of large numbers of several quantum systems are usually described with the concept of *density matrices*. In particular, couplings of (large numbers of) quantum systems to macroscopic systems can be effectively described using density matrices. It is common to call statistical mixtures of quantum systems *mixed states*.

We consider an N-dimensional complex Hilbert space $\mathscr{H}$. For example, the Hilbert space $\mathscr{H}$ could consist of the wave functions of an electron (or a more complex system) at N different levels of excitement. In the following, we identify such a Hilbert space $\mathscr{H}$ with $\mathbb{C}^N$. This means, we identify a given quantum state by its coordinates with respect to some given basis $\{\varphi_j\}_{j=1,\dots,N}$. A practical choice for a basis is the set of eigenfunctions of the Hamiltonian H of the unperturbed system (usually the quantum harmonic oscillator).

Next, we consider a statistical mixture of an arbitrary large number M of normalized states $\{\psi_j\}_{j=1,\dots,M} \in \mathbb{C}^N$. We assume that probabilities $\{p_j\}_{j=1,\dots,M}$ with

$$p_j \geq 0, \qquad \sum_{j=1}^{M} p_j = 1 \tag{1.1}$$

are given and that the state ψ_j occurs with probability p_j. Then, the statistical mixture can be described by the Hermitian density matrix $\rho \in \mathbb{C}^{N\times N}_{\text{herm}}$ defined by

$$\rho := \sum_{j=1}^{M} p_j \, \psi_j \otimes \overline{\psi}_j. \tag{1.2}$$

Besides the fact that ρ is Hermitian, it holds $\operatorname{Tr}(\rho) = 1$ and ρ is positive semi-definite, i.e. all eigenvalues are non-negative. We abbreviate the positive semi-definiteness of ρ with $\rho \geq 0$. Thus, the admissible set for density matrices for an N-dimensional complex Hilbert space is given by

$$\mathcal{R}_N := \Big\{\rho \in \mathbb{C}^{N\times N}_{\text{herm}} : \ \rho \geq 0, \ \operatorname{Tr}(\rho) = 1\Big\}. \tag{1.3}$$

The diagonal entries of a density matrix are called *populations* (of the corresponding states φ_j). The off-diagonal entries are called *coherences*. The expectation value of a Hermitian operator A in a mixed state ρ is calculated via $\langle A\rangle_\rho := \operatorname{Tr}(A\rho)$.

It is well known that the evolution of a single quantum state ψ is governed by the Schrödinger equation[1]

$$\partial_t \psi = -\frac{1}{\hbar}\mathsf{H}_{\text{sys}}\psi, \tag{1.4}$$

with the reduced Planck constant $\hbar := \frac{h}{2\pi}$. In general, the *Hamiltonian of the system* denoted with H_{sys}, is a Hermitian operator on the Hilbert space $\mathscr{H}$. In our case, where we identified the considered Hilbert space with $\mathbb{C}^N$, the operator H_{sys} can be identified with a Hermitian matrix from $\mathbb{C}^{N\times N}_{\text{herm}}$. Depending on the kind of system one wants to consider and the effects one wants to include, the Hamiltonian differs. It is straightforward to deduce an evolution equation for density matrices from the Schrödinger equation. The result is the so called *Liouville-von-Neumann equation*

$$\partial_t \rho = \frac{i}{\hbar}\big[\rho, \mathsf{H}_{\text{sys}}\big], \tag{1.5}$$

where $[A, B] := AB - BA$ denotes the commutator of the two operators (resp. matrices) A and B. We stress that the evolution governed by (1.5) leaves the set $\mathcal{R}_N$ invariant.

It is also possible to consider the density matrix of a single quantum state ψ. In this case, one speaks of a *pure state* (in contrast to mixed states). The density matrix $\rho_{\text{pu}} = \psi \otimes \overline{\psi}$ of a pure state is characterized by $\operatorname{Tr}(\rho_{\text{pu}}^2) = 1$, in contrast to mixed states, where we have $\operatorname{Tr}(\rho^2) < 1$.[2] The density matrices of pure states are (also called) *projection operators*.

The dynamics governed by (1.5) are purely Hamiltonian. This contradicts the observation of quantum systems in reality, where dissipation takes place[3]. The naïve way to account for these dissipative effects, is to modify (1.5) and add *phenomenological relax-*

[1] See for example [CTDL09, Sec. 3.2.4].
[2] See for example [CTDL09, Sec. 3.10.3 & 3.10.4].
[3] In the case of matter-light interaction, the main dissipative effects are spontaneous emission of light, collisions and vibrations in crystal lattices. See [BBR01].

ation terms in form of the matrix $Q(\rho)$. This yields the *damped Hamiltonian dynamics*

$$\partial_t \rho = \frac{i}{\hbar}\big[\rho, \mathsf{H}_{\text{sys}}\big] + Q(\rho). \tag{1.6}$$

A common choice[4] for the matrix $Q(\rho) = \big(q_{jk}(\rho)\big)_{j,k\in\{1,\ldots,N\}}$ is the following

$$q_{jk}(\rho) := -\delta_{jk} \sum_{l=1}^{N} \big(\alpha_{jl}\rho_{jj} - \alpha_{lj}\rho_{ll}\big) - \big(1-\delta_{jk}\big)\beta_{jk}\rho_{jk}. \tag{1.7}$$

Clearly, the constants $\alpha_{jj} \geq 0$ and $\beta_{jk} = \beta_{kj} \geq 0$ have the dimension of an inverse time. They are called *relaxation rates.* Another common choice[5] for the matrix $Q(\rho) = \big(q_{jk}(\rho)\big)_{j,k\in\{1,\ldots,N\}}$ is

$$q_{jk}(\rho) := -\delta_{jk}\alpha_{jj}\big(\rho_{jj} - \rho_{jj}^{\text{eq}}\big) - \big(1-\delta_{jk}\big)\beta_{jk}\rho_{jk}. \tag{1.8}$$

Here, $\alpha_{jj}^{-1} \geq 0$ are called *longitudinal relaxation times,* $\beta_{jk}^{-1} \geq 0$ are called *transversal relaxation times* and ρ_{jj}^{eq} are called *equilibrium populations.*

In order to ensure that the set $\mathcal{R}_N$ is left invariant by the evolution governed by (1.6), with $Q(\rho)$ given by (1.7) or (1.8), the relaxation rates or relaxation times as well as the equilibrium populations ρ_{jj}^{eq} cannot be chosen freely. This is certainly one of the shortcomings of the approaches above. In Section 2.4 we will take into account that dissipation takes place in a more systematic way. The result will be a self-consistent model that intrinsically ensures the invariance of $\mathcal{R}_N$.

1.1.1. Perturbation by a classical Electromagnetic Field

If a statistical mixture of quantum systems, described by the density matrix ρ, is perturbed by an incident electromagnetic wave, represented by the *electric field* $\mathbf{E}$, the Hamiltonian of the system has to be adapted. In this case we have $\mathsf{H}_{\text{sys}} = \mathsf{H} + \mathsf{H}(\mathbf{E})$ and call $\mathsf{H}(\mathbf{E})$ the *interaction Hamiltonian.* A good approximation[6] of the interaction Hamiltonian is given by the constitutive equation

$$\mathsf{H}(\mathbf{E}) := -\Gamma^*\mathbf{E}, \tag{1.9}$$

with the linear (electric) *dipole moment operator* $\Gamma^* \in \mathcal{L}(\mathbb{R}^3_{\mathbf{E}}, \mathbb{C}^{N\times N}_{\text{herm}})$. We note that the dipole moment operator is defined in terms of the *position operator* $\widehat{\mathbf{p}}$. Moreover, we say that the dipole moment operator Γ^* is (linearly) *polarized* if a unit vector $\mathbf{g} \in \mathbb{R}^3_{\mathbf{E}}$ and a Hermitian matrix $\mathsf{G}^* \in \mathbb{C}^{N\times N}_{\text{herm}}$ exist, such that

$$\forall\, \mathbf{E} \in \mathbb{R}^3_{\mathbf{E}} : \qquad \Gamma^*\mathbf{E} = \big(\mathbf{g}\cdot\mathbf{E}\big)\mathsf{G}^*. \tag{1.10}$$

[4]See for example [BBR01], [CaD12] and [GAF10, Sec. 2C.1.2].

[5]See for example [AlE87, Sec. 3.4], [PaP69, Sec. 1.4.3] and [Boy03, Sec. 6.2].

[6]In [PaP69, Sec. 2.2] and [GAF10, Sec. 2.2.3 & 2.2.4] this is explained in detail. See also [AlE87, Sec. 2.3].

Often, the dipole moment operator is interpreted as a Hermitian $N \times N$-matrix with entries in $\mathbb{C}^3$. Then, the application to $\mathbf{E}$ is to be understood in the sense that

$$\big(\Gamma^*\mathbf{E}\big)_{jk} = \Gamma^*_{jk} \cdot \mathbf{E}. \tag{1.11}$$

In particular, with this interpretation it is clear that we have $\langle \Gamma^* \rangle_\rho = \mathrm{Tr}(\Gamma^* \rho) \in \mathbb{R}^3_{\mathbf{E}}$ and we identify $\mathbf{P} := \langle \Gamma^* \rangle_\rho$ as the polarization[7] of our mixed state described by ρ. Moreover, with the above interpretation, the operator Γ defined by

$$\Gamma : \begin{cases} \mathbb{C}^{N\times N}_{\mathrm{herm}} \longrightarrow \mathbb{R}^3_{\mathbf{E}} \\ \qquad\quad \rho \longmapsto \mathrm{Tr}(\Gamma^*\rho) \end{cases} \tag{1.12}$$

is the adjoint of the dipole moment operator Γ^*. In the case of a polarized dipole moment operator Γ^*, this interpretation yields the existence of a unit vector $\mathbf{g} \in \mathbb{R}^3_{\mathbf{E}}$ and a Hermitian matrix $\mathsf{G}^* \in \mathbb{C}^{N\times N}_{\mathrm{herm}}$, satisfying $\big(\Gamma^*\big)_{jk} = \big(\mathsf{G}^*\big)_{jk}\mathbf{g}$. In particular, for the polarization we get $\mathbf{P} = \mathrm{Tr}\big(\mathsf{G}^*\rho\big)\mathbf{g}$.

1.1.2. The Two-Level Case

Next, we take a closer look at the situation where a mixture of two quantum systems is perturbed by an incident electromagnetic field represented by the electric field $\mathbf{E}$. We will refer to this case as the *two-level case.*[8] We assume that the basis of the considered 2-dimensional Hilbert space $\mathscr{H}$ is actually given by the set of eigenfunctions of the unperturbed Hamiltonian H, and that the dipole moment operator Γ^* is polarized. Then, after identification, H is a diagonal matrix, and it is reasonable[9] to assume that G^* has entries in the off-diagonals, only. Therefore, we fix

$$\mathsf{G}^* = \begin{pmatrix} 0 & \gamma_{12} \\ \gamma_{21} & 0 \end{pmatrix}, \qquad \mathsf{H} = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}, \qquad \rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \tag{1.13}$$

with $\gamma_{21} = \overline{\gamma}_{12}$ and $\omega_2 > \omega_1 > 0$. For a given incident electric field $\mathbf{E}$, we set $\mathrm{e} := \mathbf{g} \cdot \mathbf{E} \in \mathbb{R}_{\mathbf{E}}$ and consider the damped Hamiltonian dynamics governed by

$$\partial_t \rho = \frac{i}{\hbar}\big[\rho, \mathsf{H} - \mathrm{e}\mathsf{G}^*\big] + Q(\rho), \tag{1.14}$$

where $Q(\rho)$ is chosen as in (1.8). For the relaxation times we assume $\alpha_{11}^{-1} = \alpha_{22}^{-1} =: T_1$ and $\beta_{12}^{-1} = \beta_{21}^{-1} =: T_2$, and for the equilibrium populations we assume $\rho_{11}^{\mathrm{eq}} + \rho_{22}^{\mathrm{eq}} = 1$. Then, with $\omega_{21} := \omega_2 - \omega_1$ and $j := i/\hbar$, we get the following system of ordinary differential

[7] In fact, this is the other usual constitutive equation of semi-classical matter-light interaction.

[8] This situation is well studied in the literature. See for example [PaP69, Sec. 2.4], [Boy03, Sec. 6.2], [AlE87, Sec. 2.3 & Sec. 3.4] and [GAF10, Sec. 2C.3.1] where no field is present.

[9] This has to do with the dependence of Γ^* on the position operator $\widehat{\mathbf{p}}$ and the assumption that the eigenfunctions of H have definite parities, i.e. the eigenfunctions are either even or odd. For more details, we refer to [GAF10, comment on p. 61 & Section 2B], [JMR00b, Sec. 12] and [PaP69, Sec. 2.3].

equations

$$\partial_t \rho_{11} = j\mathrm{e}\big(\gamma_{12}\rho_{21} - \gamma_{21}\rho_{12}\big) - \tfrac{1}{T_1}\big(\rho_{11} - \rho_{11}^{\mathrm{eq}}\big) \tag{1.15a}$$
$$\partial_t \rho_{12} = j\omega_{21}\rho_{12} + j\mathrm{e}\gamma_{12}(\rho_{22} - \rho_{11}) - \tfrac{1}{T_2}\rho_{12} \tag{1.15b}$$
$$\partial_t \rho_{21} = -j\omega_{21}\rho_{21} - j\mathrm{e}\gamma_{21}(\rho_{22} - \rho_{11}) - \tfrac{1}{T_2}\rho_{21} \tag{1.15c}$$
$$\partial_t \rho_{22} = -j\mathrm{e}\big(\gamma_{12}\rho_{21} - \gamma_{21}\rho_{12}\big) - \tfrac{1}{T_1}\big(\rho_{22} - \rho_{22}^{\mathrm{eq}}\big). \tag{1.15d}$$

Introducing the alternative real variables

$$w := \rho_{22} - \rho_{11}, \qquad p := \gamma_{12}\rho_{21} + \gamma_{21}\rho_{12}, \qquad q := -j\big(\gamma_{12}\rho_{21} - \gamma_{21}\rho_{12}\big), \tag{1.16}$$

we get the following set of equivalent[10] equations

$$\partial_t w = 2\mathrm{e}q - \tfrac{1}{T_1}(w - w_{\mathrm{eq}}) \tag{1.17a}$$
$$\partial_t p = \omega_{21} q - \tfrac{1}{T_2} p \tag{1.17b}$$
$$\partial_t q = -\tfrac{1}{\hbar^2}\omega_{21} p - 2\tfrac{1}{\hbar^2}|\gamma_{12}|^2 \mathrm{e} w - \tfrac{1}{T_2} q. \tag{1.17c}$$

Of course, also the Hamiltonian dynamics from (1.5) can be described in the real variables p, q, w. In this case one gets the equations (1.17) without the relaxation terms $\frac{1}{T_2}q$, $\frac{1}{T_2}p$ and $\frac{1}{T_1}(w - w_{\mathrm{eq}})$. We will refer to this case as the *undamped version of* (1.17).

1.2. Classical Maxwell-Bloch Equations

In this section we take a first approach to the semi-classical modeling of matter-light interaction and derive the classical Maxwell-Bloch equations. We consider an (optically) *active material* (a two-level quantum system) described by the density matrix $\rho \in \mathbb{C}^{2\times 2}_{\mathrm{herm}}$ which is coupled to an electromagnetic field[11] $\big(\mathbf{E}, \mathbf{H}\big) \in \mathbb{R}^3_{\mathbf{E}} \times \mathbb{R}^3_{\mathbf{H}}$ describing light.

The evolution of the electromagnetic field $\big(\mathbf{E}, \mathbf{H}\big)$ is governed by the time dependent Maxwell equations for non-magnetizable materials[12]

$$\epsilon_0 \partial_t \mathbf{E} - \operatorname{curl} \mathbf{H} = -\partial_t \mathbf{P} - \mathbf{j} - \sigma \mathbf{E} \tag{1.18a}$$
$$\mu_0 \partial_t \mathbf{H} + \operatorname{curl} \mathbf{E} = 0 \tag{1.18b}$$

with a *prescribed external current* $\mathbf{j}$, the *conductivity* σ, the *electric permittivity* ϵ_0 and the *magnetic permeability* μ_0.[13] Physical solutions to Maxwell's equations in the absence of the current $\mathbf{j} + \sigma\mathbf{E}$ and free charges are those satisfying

$$\operatorname{div}\big(\epsilon_0 \mathbf{E} + \mathbf{P}\big) = 0, \qquad \operatorname{div} \mathbf{H} = 0. \tag{1.19}$$

[10]The equivalence is subject to choosing initial values from $\mathcal{R}_2$.
[11]The electric field $\mathbf{E}$ and the *magnetic field* $\mathbf{H}$.
[12]See for example [Jac06, Sec. 6.6].
[13]Here, the total amount of free currents is $\mathbf{j}+\sigma\mathbf{E}$, but one distinguishes between the prescribed external current $\mathbf{j}$ and the induced current $\sigma\mathbf{E}$. Most often, both quantities are assumed to be zero.

At least formally these conditions are satisfied for all times, if they are satisfied initially.[14] First, we study the case where the evolution of the density matrix is governed by the Hamiltonian dynamics (1.5). In order to make sense of the coupling of the space-time dependent Maxwell equations (1.18) and the purely local equation (1.5), we assume that ρ and H parametrically depend on x.[15]

For the coupling we assume that on the one hand, the electromagnetic field responds to the polarization of the active material given by the constitutive equation $\mathbf{P} = \mathrm{Tr}(\mathsf{G}^*\rho)\mathbf{g}$ involving the polarized dipole moment operator Γ^* expressed by $\mathbf{g} \in \mathbb{R}^3_{\mathbf{E}}$ and $\mathsf{G}^* \in \mathbb{C}^{2\times 2}_{\mathrm{herm}}$. On the other hand, we assume that the active material responds to the electromagnetic field via the interaction Hamiltonian $\mathsf{H}(\mathbf{E})$ from (1.9). Thus, the system Hamiltonian under consideration is $\mathsf{H}_{\mathrm{sys}} = \mathsf{H} - \big(\mathbf{g}\cdot\mathbf{E}\big)\mathsf{G}^*$. Clearly, due to our considerations for the two-level case and recalling that $\mathrm{e} := \mathbf{g}\cdot\mathbf{E}$, we can equivalently describe the evolution of ρ in the variables w, p, q from (1.16).

In the following, we assume that the unperturbed Hamiltonian H and the matrix G^* are given as in (1.13) and we recall that $\gamma_{21} = \overline{\gamma}_{12}$ and $\omega_{21} := \omega_2 - \omega_1 > 0$. Under these assumptions, we get the following expression for the polarization

$$\mathbf{P} = \mathrm{Tr}(\Gamma^*\rho) = \mathrm{Tr}(\mathsf{G}^*\rho)\mathbf{g} = \mathrm{Tr}\begin{pmatrix}\gamma_{12}\rho_{21} & \gamma_{12}\rho_{22}\\ \gamma_{21}\rho_{12} & \gamma_{21}\rho_{11}\end{pmatrix}\mathbf{g} = \big(\gamma_{12}\rho_{21} + \gamma_{21}\rho_{12}\big)\mathbf{g}. \tag{1.20}$$

In the case of Hamiltonian dynamics, we can infer from the identity $\mathrm{Tr}(A[A,B]) = 0$ (see Lemma A.1.19) that for the evolution equation of the polarization we have

$$\begin{aligned}\partial_t\mathbf{P} &= \mathrm{Tr}\Big(\Gamma^* j\big[\rho,\, \mathsf{H} - \Gamma^*\mathbf{E}\big]\Big) = -j\mathbf{g}\,\mathrm{Tr}\Big(\mathsf{G}^*\big[\mathsf{H},\,\rho\big]\Big) + j\mathbf{g}\,\mathrm{Tr}\Big(\mathsf{G}^*\big[(\mathbf{g}\cdot\mathbf{E})\mathsf{G}^*,\,\rho\big]\Big)\\ &= -j\mathbf{g}\,\mathrm{Tr}\Big(\mathsf{G}^*\big[\mathsf{H},\,\rho\big]\Big) + j\mathbf{g}(\mathbf{g}\cdot\mathbf{E})\mathrm{Tr}\Big(\mathsf{G}^*\big[\mathsf{G}^*,\,\rho\big]\Big)\\ &= -j\mathrm{Tr}\Big(\mathsf{G}^*\big[\mathsf{H},\,\rho\big]\Big)\mathbf{g} = -j\omega_{21}\big(\gamma_{12}\rho_{21} - \gamma_{21}\rho_{12}\big)\mathbf{g}.\end{aligned} \tag{1.21}$$

Recalling the definition of the variables p and q, this implies $\mathbf{P} = p\,\mathbf{g}$ and $\partial_t\mathbf{P} = \omega_{21}q\,\mathbf{g}$. In particular, by inserting the undamped versions of (1.17b) and (1.17c) we get the following equation for $\partial_t^2\mathbf{P} = \omega_{21}\partial_t q\,\mathbf{g}$

$$\partial_t^2\mathbf{P} = -\frac{\omega_{21}^2}{\hbar^2}\mathbf{P} - \frac{2\omega_{21}}{\hbar^2}|\gamma_{12}|^2 w\mathrm{e}\mathbf{g}. \tag{1.22}$$

For the function w, the *inversion*, we get the following evolution equation from the undamped version of (1.17a) by replacing q with $\omega_{21}^{-1}\partial_t\mathbf{P}\cdot\mathbf{g}$

$$\partial_t w = \frac{2\mathrm{e}}{\omega_{21}}\partial_t\mathbf{P}\cdot\mathbf{g}. \tag{1.23}$$

[14] A short explanation is given in Chapter 4.

[15] To be precise, in this case ρ and H have to be understood as densities. The same holds for the dipole moment operator Γ^*. However, we will not use the word density, but speak of the corresponding quantities instead.

Usually, it is assumed that $\mathbf{E}$ and $\mathbf{g}$ are parallel at all times, i.e. the response of the active material depends on the direction of the electric field, but is otherwise isotropic. In this case we have $\mathbf{E} = \mathrm{e}\mathbf{g}$ and the above equations yield

$$\partial_t^2 \mathbf{P} + \omega_{21}^2 \hbar^{-2} \mathbf{P} = -2\omega_{21}\hbar^{-2}|\gamma_{12}|^2 w\mathbf{E} \tag{1.24a}$$

$$\partial_t w = \frac{2}{\omega_{21}} \mathbf{E} \cdot \partial_t \mathbf{P}. \tag{1.24b}$$

Thus, in the undamped case, we can self-consistently reduce the study of the five equations (1.17) & (1.18), to the study of the four equations (1.18) & (1.24).

Next, we study the case where the evolution of the density matrix in the variables (w, p, q) is governed by the damped Hamiltonian dynamics (1.17). In this case we have $\mathbf{P} = p\,\mathbf{g}$ and $\partial_t \mathbf{P} = (\omega_{21} q - \frac{1}{T_2} p)\mathbf{g}$. In particular, this implies

$$\partial_t^2 \mathbf{P} = \big(\omega_{21}\partial_t q - \tfrac{1}{T_2}\partial_t p\big)\mathbf{g} \qquad \text{and} \qquad q = \tfrac{1}{\omega_{21}}\big(\partial_t \mathbf{P}\cdot\mathbf{g} + \tfrac{1}{T_2} p\big). \tag{1.25}$$

Insertion of (the damped versions of) (1.17b) and (1.17c) into the equation for $\partial_t^2 \mathbf{P}$ yields

$$\partial_t^2 \mathbf{P} = -\frac{\omega_{21}^2}{\hbar^2}\mathbf{P} - \frac{2\omega_{21}}{\hbar^2}|\gamma_{12}|^2 w\mathrm{e}\mathbf{g} - \frac{2}{T_2}\partial_t \mathbf{P} - \frac{1}{T_2^2}\mathbf{P}. \tag{1.26}$$

Inserting $q = \frac{1}{\omega_{21}}\big(\partial_t \mathbf{P}\cdot\mathbf{g} + \frac{1}{T_2} p\big)$ into (the damped version of) equation (1.17a) yields

$$\partial_t w = \frac{2\mathrm{e}}{\omega_{21}}\big(\partial_t \mathbf{P}\cdot\mathbf{g} + \tfrac{1}{T_2} p\big) - \frac{1}{T_1}(w - w_{\mathrm{eq}}). \tag{1.27}$$

In the following, we drop the terms $T_2^{-2}\mathbf{P}$ in (1.26) and $T_2^{-1}p$ in (1.27).[16] As before, if $\mathbf{E}$ and $\mathbf{g}$ are parallel at all times, the above equations yield

$$\partial_t^2 \mathbf{P} + \frac{2}{T_2}\partial_t \mathbf{P} + \frac{\omega_{21}^2}{\hbar^2}\mathbf{P} = -\frac{2\omega_{21}}{\hbar^2}|\gamma_{12}|^2 w\mathbf{E} \tag{1.28a}$$

$$\partial_t w = \frac{2}{\omega_{21}}\mathbf{E}\cdot\partial_t \mathbf{P} - \frac{1}{T_1}(w - w_{\mathrm{eq}}). \tag{1.28b}$$

A similar and more comprehensive derivation of this system can be found in the monograph [PaP69, Sec. 2.4]. We call the coupled system (1.18) & (1.28) for the unknown functions $\big(\mathbf{E}, \mathbf{H}, \mathbf{P}, w\big)$ the *classical Maxwell-Bloch system.* Furthermore, for an active material consisting of a mixture of an arbitrary (but finite) number N of quantum states, we will call the system (1.18) & (1.6) for the unknown functions $\big(\mathbf{E}, \mathbf{H}, \rho\big)$ with the above coupling mechanism the *classical full Maxwell-Bloch system.* Next, we will give an overview of the literature, where both the classical Maxwell-Bloch system and the classical full Maxwell-Bloch system have been studied (mathematically).

[16] A physical justification can be found in [PaP69] Sec. 2.4.2, p. 33 and Sec. 2.4.4, p. 36.

1.3. State of the Art Maxwell-Bloch Analysis

The first approach of a mathematical analysis for the classical Maxwell-Bloch system without the currents $\mathbf{j}$ and $\sigma\mathbf{E}$ was done by P. Donnat and J. Rauch in [DoR96]. They proved global existence for smooth initial data from H^s, $s \geq 2$. F. Jochmann also considered the classical Maxwell-Bloch system without the currents but with $T_1 = \infty$ in [Joc03]. There, he also analyzed the long time behavior of the system. Furthermore, for the classical Maxwell-Bloch system with currents, he proved convergence to stationary states in [Joc02a].

The first attempt on analyzing the classical full Maxwell-Bloch system is due to É. Dumas. Based on the article [JMR00a] of J. L. Joly, G. Metivier and J. Rauch, where the related Landau-Lifshitz equations have been studied, É. Dumas proved in [Dum05] global existence of weak L^2-solutions to initial value problems for a class of systems. This class of systems includes the classical full Maxwell-Bloch system without currents and with constant ϵ_0, $\mu_0 > 0$. In particular, É. Dumas proved[17] that the solutions are unique if the initial data of the fields has curl in L^2.

With an improvement of the methods from [JMR00a], F. Jochmann showed in the article [Joc02b] global existence for a version of the Landau-Lifshitz equations similar to the one considered in [JMR00a]. F. Jochmann could handle variable ϵ_0, $\mu_0 \in L^\infty$, in contrast to the former one and his model included the currents $\mathbf{j}$ and $\sigma\mathbf{E}$.

Based on F. Jochmanns article [Joc02b], É. Dumas and F. Sueur showed in [DuS12] existence of solutions to initial value problems for a class of systems including the full Maxwell-Bloch system without currents, but with variable, uniformly positive functions ϵ_0, $\mu_0 \in L^\infty$. For smooth functions ϵ_0, μ_0 and smooth initial data they also showed uniqueness.

Finally, in his article [Joc05] F. Jochmann studied the long time behavior of the version of the Landau-Lifshitz equations he considered in [Joc02b].

It should also be mentioned that the coupling of (1.24) with the spacial one-dimensional version of (1.18) was considered in [JMR00b, Ex. 12.2]. Moreover, in [Joc07] the long time behavior of the coupling of (1.24) with (1.18) involving the current $\sigma\mathbf{E}$, only, was studied.

1.4. A Reduced Model

In an electromagnetic field $(\mathbf{E}, \mathbf{H})$ in vacuum, the electric field $\mathbf{E}$ is much stronger than the magnetic field $\mathbf{H}$.[18] This is certainly one of the reasons why one is interested in a description of the coupling of an active material with an electric field, only.

In the following, we derive a reduced Maxwell-Bloch type system where the *amplitude* of an electric field is coupled to the inversion of a two-level system. We emphasize that it is not clear whether or not our assumptions can be physically justified.[19]

[17] His proof contains a mistake. See Section 4.4 for further details.
[18] In fact, we have $|\mathbf{E}| = c_0\mu_0^{-1}|\mathbf{H}|$ with the *vacuum speed of light* c_0, see [Dem13, Sec. 7.5].
[19] It may be that this approach is a meaningless heuristic.

We assume that for all times

$$\operatorname{div}\big(\epsilon_0\mathbf{E}+\mathbf{P}\big)=0,\qquad \operatorname{div}\mathbf{H}=0 \tag{1.19}$$

is satisfied and that the active material is a *linear isotropic medium*, i.e. we assume that with the *electric permittivity of the medium* ϵ_r it holds $\epsilon_0\mathbf{E}+\mathbf{P}=\epsilon_r\epsilon_0\mathbf{E}$. Furthermore, we assume $\mathbf{j}+\sigma\mathbf{E}=0$. Then, applying ∂_t to (1.18a) and curl to (1.18b) yields

$$\epsilon_0\partial_t^2\mathbf{E}-\partial_t\operatorname{curl}\mathbf{H}=-\partial_t^2\mathbf{P},\qquad \mu_0\,\partial_t\operatorname{curl}\mathbf{H}+\operatorname{curl}\operatorname{curl}\mathbf{E}=0. \tag{1.29}$$

From (1.19) we get $\operatorname{curl}\operatorname{curl}\mathbf{E}=\nabla\operatorname{div}\mathbf{E}-\Delta\mathbf{E}=-\Delta\mathbf{E}$. We identify the quantity $(\epsilon_0\mu_0)^{-1/2}=c_0$ as the vacuum speed of light. Thus, we get the following inhomogeneous wave equation for the electric field $\mathbf{E}$

$$\Delta\mathbf{E}-\frac{1}{c_0^2}\partial_t^2\mathbf{E}=\mu_0\partial_t^2\mathbf{P}. \tag{1.30}$$

Next, we couple equation (1.30) for $\mathbf{E}$ with system (1.17) via $\mathrm{e}:=\mathbf{g}\cdot\mathbf{E}$. We recall[20]

$$w:=\rho_{22}-\rho_{11},\qquad p:=\gamma_{12}\rho_{21}+\gamma_{21}\rho_{12},\qquad q:=-j\big(\gamma_{12}\rho_{21}-\gamma_{21}\rho_{12}\big),$$

from (1.16). Due to the isotropy of the medium, the unit vector $\mathbf{g}\in\mathbb{R}^3_{\mathbf{E}}$ is parallel to $\mathbf{E}$ at all times. Moreover, we assume that for $\mathbf{P}$ we have[21]

$$\partial_t^2\mathbf{P}=-\omega_{21}\big(\tfrac{1}{\hbar^2}\omega_{21}p+2\tfrac{1}{\hbar^2}|\gamma_{12}|^2(\mathbf{g}\cdot\mathbf{E})w\big)\mathbf{g}, \tag{1.31}$$

but that the evolution of the variables $(w,p,q,)$ is governed by (1.17) which originates from the damped Hamiltonian dynamics (1.6). Namely, we consider the following system

$$\begin{aligned}
\Delta\mathbf{E}-c_0^{-2}\partial_t^2\mathbf{E}&=\mu_0\partial_t^2\mathbf{P} &&(1.32\text{a})\\
\partial_t^2\mathbf{P}&=-\omega_{21}\big(\tfrac{1}{\hbar^2}\omega_{21}p\,\mathbf{g}+2\tfrac{1}{\hbar^2}|\gamma_{12}|^2\mathbf{E}\,w\big) &&(1.32\text{b})\\
\partial_t w&=2(\mathbf{g}\cdot\mathbf{E})q-\tfrac{1}{T_1}(w-w_{\mathrm{eq}}) &&(1.32\text{c})\\
\partial_t p&=\omega_{21}q-\tfrac{1}{T_2}p &&(1.32\text{d})\\
\partial_t q&=-\tfrac{1}{\hbar^2}\omega_{21}p-2\tfrac{1}{\hbar^2}|\gamma_{12}|^2(\mathbf{g}\cdot\mathbf{E})w-\tfrac{1}{T_2}q. &&(1.32\text{e})
\end{aligned}$$

Next, we make some simplifying assumptions to eliminate the variables p and q. We make the ansatz

$$\rho_{12}(x,t)=\widehat{\rho}(x,t)\cdot\exp(j\omega_{21}t),\qquad \rho_{21}(x,t)=\widehat{\rho}(x,t)\cdot\exp(-j\omega_{21}t) \tag{1.33}$$

and assume that it holds $|\partial_t\widehat{\rho}|\ll|j\omega_{21}|$. Moreover, we perform a *slowly varying envelope*

[20] As before, we assume that the densities w, p, q parametrically depend on x.

[21] This equation results from the definition $\mathbf{P}:=\operatorname{Tr}\big(\mathsf{G}^*\rho\big)\mathbf{g}$, but assuming Hamiltonian dynamics as in the beginning of Section 1.2. This approach is of course not self-consistent.

approximation. This means, we assume

$$\partial_t \rho_{12}(x,t) \approx j\omega_{21}\widehat{\rho}(x,t)\cdot\exp(j\omega_{21}t), \qquad \partial_t \rho_{21}(x,t) \approx -j\omega_{21}\widehat{\rho}(x,t)\cdot\exp(-j\omega_{21}t). \tag{1.34}$$

This assumption yields

$$\partial_t p \approx \omega_{21} q, \qquad \text{thus} \qquad p \approx 0 \tag{1.35a}$$

$$\partial_t q \approx -\frac{1}{\hbar^2}\omega_{21} p, \qquad \text{thus} \qquad q \approx -\frac{2T_2}{\hbar^2}|\gamma_{12}|^2(\mathbf{g}\cdot\mathbf{E})w. \tag{1.35b}$$

Furthermore, we assume that the electric field is a unidirectional plane wave and make the following ansatz

$$\mathbf{E}(x,t) = a(x,t)\exp(-j(\mathbf{k}\cdot x - \omega_{21}t))\mathbf{g}. \tag{1.36}$$

Here, we assume that a is a complex amplitude and that the wave vector $\mathbf{k}$ satisfies $|\mathbf{k}| = \omega_{21}/c_0$ and $\mathbf{k} \parallel \mathbf{g}$. We also perform a slowly varying envelope approximation for the electric field, i.e. we assume

$$|\partial_t^2 a| \ll |j\omega_{21}\partial_t a| \qquad \text{and} \qquad |\Delta a| \ll |\mathbf{k}|\,|\nabla_{\mathbf{g}} a|. \tag{1.37}$$

From this we get the following approximation of the d'Alembert operator (using that $\omega_{21} = |\mathbf{k}|c_0$)

$$\Big(\Delta - \frac{1}{c_0^2}\partial_t^2\Big)\mathbf{E}(x,t) \approx -2j\mathbf{k}\exp(j(\omega_{21}t - \mathbf{k}\cdot x))\Big(\nabla_{\mathbf{g}} + \frac{1}{c_0}\partial_t\Big)a(x,t). \tag{1.38}$$

Inserting (1.35a), (1.35b), (1.36) and (1.38) into (1.32a)–(1.32c) yields the following reduced system[22] for the amplitude a and the inversion w

$$\Big(\nabla_{\mathbf{g}} + \frac{1}{c_0}\partial_t\Big)a(x,t) = \frac{-i\mu_0 c_0}{\hbar}|\gamma_{12}|^2 a(x,t)\,w(x,t) \tag{1.39a}$$

$$\partial_t w(x,t) = -\frac{4T_2}{\hbar^2}|\gamma_{12}|^2|a(x,t)|^2 w(x,t) - \frac{1}{T_1}(w(x,t) - w_{\mathrm{eq}}(x)). \tag{1.39b}$$

We will refer to this system as the *reduced unidirectional Maxwell-Bloch system.* In Chapter 2 we will abstract system (1.39) before, in Chapter 3, we will perform a mathematical study of the resulting abstracted system, including an analysis of its long time behavior.

[22] In [Kae05, Ch. 2], a similar reduced model has been derived. See also [Boy03, Sec. 13.2]. Mathematical studies of other reduced models can for example be found in [JoR02] and [LRR07].

2. Isothermal Damped Hamiltonian Systems in the GENERIC Framework

In this chapter we state our concept of isothermal damped Hamiltonian systems in the GENERIC[1] framework and introduce some examples.[2] All examples are related to semi-classical matter-light interaction. A highlight of this chapter is the introduction of an evolution equation that consistently[3] describes the dissipative dynamics of a density matrix ρ in Section 2.4 and its coupling to Maxwell's equations in Section 2.5. In the forthcoming Chapter 3 and Chapter 4 our focus lies on the two systems resulting from Section 2.2 and Section 2.5. In both cases a symmetric hyperbolic system for $\mathbf{u}$ is coupled to an ODE for $\mathbf{v}$ depending parametrically on the space variable x

$$\partial_t \mathbf{u}(x,t) + \sum_{j=1}^{d} A_j \partial_{x_j} \mathbf{u}(x,t) = \mathrm{F}(x, \mathbf{v},), \qquad \partial_t \mathbf{v}(x,t) = \mathrm{G}(x, \mathbf{u}, \mathbf{v}).$$

The non-linearities G for the two systems are rather similar in the variable $\mathbf{v}$. An interesting difference is that in one case G is linear in $\mathbf{u}$ and in the other case, G is linear in $|\mathbf{u}|^2$. In Chapter 3 and Chapter 4 we will perform a thorough analysis of these systems.

2.1. Mathematical Framework

We start with giving our general concept of an *isothermal damped Hamiltonian system* in the GENERIC framework[4]. Let $\mathcal{X}$ be some given state space and let $\vartheta_* > 0$ be a fixed temperature. For a given $u \in \mathcal{X}$ let $T_u\mathcal{X}$ denote the tangent space at u and let $T_u^*\mathcal{X} = (T_u\mathcal{X})^*$ denote its dual, called the cotangent space. The dual pairing between $v \in T_u\mathcal{X}$ and $\xi \in T_u^*\mathcal{X}$ is denoted by $\langle \xi, v \rangle_{T_u\mathcal{X}}$ and the tangent bundle $\bigcup_{u\in\mathcal{X}}(u, T_u\mathcal{X})$ is abbreviated by $T\mathcal{X}$.

We give the following formal definition of a damped Hamiltonian system where we assume that for each $u \in \mathcal{X}$ the *derivative of the isothermal free energy functional*

[1] GENERIC stands for General Equations for Non-Equilibrium Reversible Irreversible Coupling.

[2] For more information on GENERIC systems we refer to [Mie11] and [MiT].

[3] By consistently we mean that the admissible set for density matrices is left invariant by the evolution.

[4] In the following we will use the term *damped Hamiltonian system* and *isothermal damped Hamiltonian system* interchangeably.

$D\mathcal{F}_{\vartheta_*}(u) \in T_u^*\mathcal{X}$ is well defined and that the *isothermal free energy functional* $\mathcal{F}_{\vartheta_*}$ satisfies the chain rule $\frac{d}{dt}\mathcal{F}_{\vartheta_*}(u(t)) = \langle D\mathcal{F}_{\vartheta_*}(u(t)),\, \partial_t u(t)\rangle_{T_{u(t)}\mathcal{X}}$. In particular, we assume that the same holds for the *energy functional* $\mathcal{E}$ and the *entropy functional* $\mathcal{S}$.

Definition 2.1.1 (Damped Hamiltonian System). *A damped Hamiltonian system is the evolution quadrupel* $(\mathcal{X}, \mathcal{F}_{\vartheta_*}, \mathbb{J}, \mathbb{K})$ *consisting of a Banach space* $\mathcal{X}$*, an isothermal free energy functional* $\mathcal{F}_{\vartheta_*} : \mathcal{X} \longrightarrow \mathbb{R}$ *at temperature* ϑ_* *as well as a Poisson operator* $\mathbb{J}(u) : T_u^*\mathcal{X} \longrightarrow T_u\mathcal{X}$ *and an Onsager operator* $\mathbb{K}(u) : T_u^*\mathcal{X} \longrightarrow T_u\mathcal{X}$*. The isothermal free energy functional* $\mathcal{F}_{\vartheta_*}$ *is defined as the sum* $\mathcal{F}_{\vartheta_*} := \mathcal{E} - \frac{1}{\vartheta_*}\mathcal{S}$ *with an energy functional* $\mathcal{E} : \mathcal{X} \longrightarrow \mathbb{R}$ *and an entropy functional* $\mathcal{S} : \mathcal{X} \longrightarrow \mathbb{R}$*. The Poisson and Onsager operators are linear and satisfy the following symmetry conditions*

$$\forall\, u \in \mathcal{X} : \qquad \mathbb{J}(u) = -\mathbb{J}(u)^*, \qquad \mathbb{K}(u) = \mathbb{K}(u)^* \tag{2.1}$$

and the following structural conditions

$$\forall\, u \in \mathcal{X} : \qquad \mathbb{J}(u) \textit{ satisfies Jacobi's identity,} \tag{2.2a}$$
$$\forall\, u \in \mathcal{X} : \qquad \mathbb{K}(u) \textit{ is positive semi-definite.} \tag{2.2b}$$

The evolution for $u \in \mathcal{X}$ *is governed by the following equation in the space* $T\mathcal{X}$

$$\partial_t u = \Big(\mathbb{J}(u) - \frac{1}{\vartheta_*}\mathbb{K}(u)\Big) D\mathcal{F}_{\vartheta_*}(u). \tag{2.3}$$

Remark 2.1.2. *As remarked in [Mie11], our definition of damped Hamiltonian systems is closely related to the so called metriplectic systems introduced in [Kau84] and [Mor86]. In fact, GENERIC systems have their origins in metriplectic systems. As noted in [Mie15] an outline of the early developments can be found in [BMR13]. The name GENERIC was introduced by Öttinger and Grmela in [GrÖ97] and [ÖtG97].*

For all $u \in \mathcal{X}$ the operators $\mathbb{J}^*(u)$ and $\mathbb{K}^*(u)$ map the space $(T_u\mathcal{X})^*$ into the space $(T_u\mathcal{X})^{**}$. The symmetries (2.1) are to be understood in the sense that for all $u \in \mathcal{X}$ and for all $\xi,\, \eta \in T_u^*\mathcal{X}$ we have

$$\big\langle \mathbb{J}^*(u)\xi,\, \eta\big\rangle_{T_u^*\mathcal{X}} = -\big\langle \eta,\, \mathbb{J}(u)\xi\big\rangle_{T_u\mathcal{X}}, \qquad \big\langle \mathbb{K}^*(u)\xi,\, \eta\big\rangle_{T_u^*\mathcal{X}} = \big\langle \eta,\, \mathbb{K}(u)\xi\big\rangle_{T_u\mathcal{X}}. \tag{2.4}$$

Jacobi's identity for $\mathbb{J}$ means that for all functionals $\mathcal{F}_j : \mathcal{X} \longrightarrow \mathbb{R}$, $j = 1, 2, 3$, the identity

$$\big\{\{\mathcal{F}_1, \mathcal{F}_2\}_{\mathbb{J}}, \mathcal{F}_3\big\}_{\mathbb{J}} + \big\{\{\mathcal{F}_2, \mathcal{F}_3\}_{\mathbb{J}}, \mathcal{F}_1\big\}_{\mathbb{J}} + \big\{\{\mathcal{F}_3, \mathcal{F}_1\}_{\mathbb{J}}, \mathcal{F}_2\big\}_{\mathbb{J}} \equiv 0 \tag{2.5}$$

is satisfied, where the Poisson bracket is defined by

$$\{\mathcal{F}, \mathcal{G}\}_{\mathbb{J}}(u) := \big\langle D\mathcal{F}(u),\, \mathbb{J}(u) D\mathcal{G}(u)\big\rangle_{T_u\mathcal{X}}. \tag{2.6}$$

Positive semi-definiteness for the operator $\mathbb{K}$ means that for all $u \in \mathcal{X}$ and for all $\xi \in T_u^*\mathcal{X}$ it holds

$$\big\langle \xi,\, \mathbb{K}(u)\xi\big\rangle_{T_u\mathcal{X}} \geq 0. \tag{2.7}$$

A consequence of the skew symmetry of the Poisson operator is that for every $u \in \mathcal{X}$ and for every $\xi \in T_u^*\mathcal{X}$ it holds

$$\big\langle \xi,\, \mathbb{J}(u)\xi \big\rangle_{T_u\mathcal{X}} = -\big\langle \mathbb{J}(u)^*\xi,\, \xi \big\rangle_{T_u^*\mathcal{X}} = -\big\langle \xi,\, \mathbb{J}(u)\xi \big\rangle_{T_u\mathcal{X}} = 0. \tag{2.8}$$

In contrast to isothermal damped Hamiltonian systems, GENERIC systems are thermodynamically consistent evolution quintuples. For completeness, we give a precise definition.[5]

Definition 2.1.3. *A GENERIC system is the evolution quintuple* $(\mathcal{X}, \mathcal{E}, \mathcal{S}, \mathbb{J}, \mathbb{K})$ *consisting of a Banach space* $\mathcal{X}$*, an energy functional* $\mathcal{E} : \mathcal{X} \longrightarrow \mathbb{R}$ *and an entropy functional* $\mathcal{S} : \mathcal{X} \longrightarrow \mathbb{R}$ *as well as a Poisson operator* $\mathbb{J}(u) : T_u^*\mathcal{X} \longrightarrow T_u\mathcal{X}$ *and an Onsager operator* $\mathbb{K}(u) : T_u^*\mathcal{X} \longrightarrow T_u\mathcal{X}$*. The Poisson and Onsager operator are linear and satisfy the conditions* (2.1)–(2.2)*. Moreover, the following non-interaction condition holds*

$$\forall\, u \in \mathcal{X} : \qquad \mathbb{J}(u)D\mathcal{S}(u) = 0 \qquad \textit{and} \qquad \mathbb{K}(u)D\mathcal{E}(u) = 0. \tag{NIC}$$

The evolution for $u \in \mathcal{X}$ *is governed by the following equation in the space* $T\mathcal{X}$

$$\partial_t u = \mathbb{J}(u)D\mathcal{E}(u) + \mathbb{K}(u)D\mathcal{S}(u). \tag{2.9}$$

By an adiabatic limit procedure one can derive an isothermal damped Hamiltonian system from a given GENERIC system.[6] We stress that the non-interaction condition (NIC) holds for GENERIC systems but does not necessarily hold for the energy and entropy functionals $\mathcal{E}$ and $\mathcal{S}$ building the free energy functional $\mathcal{F}_{\vartheta_*}$ in damped Hamiltonian systems. However, all examples of damped Hamiltonian systems considered in this work satisfy

$$\forall\, u \in \mathcal{X} : \qquad \mathbb{J}(u)D\mathcal{S}(u) = 0. \tag{2.10}$$

Of course, the following lemma holds for both damped Hamiltonian systems and GENERIC systems.

Lemma 2.1.4. *If the Poisson operator* $\mathbb{J}$ *is independent of* $u \in \mathcal{X}$ *and satisfies the symmetry condition* (2.1)*, Jacobi's identity* (2.5) *is automatically satisfied. Moreover, this implies that Jacobi's identity* (2.5) *is satisfied iff for all* η_1*,* η_2*,* $\eta_3 \in T_u^*\mathcal{X}$ *we have*

$$\Big\langle \eta_1, D_u\mathbb{J}(u)\big[\mathbb{J}\eta_3, \eta_2\big] \Big\rangle_{T_u\mathcal{X}} + \Big\langle \eta_2, D_u\mathbb{J}(u)\big[\mathbb{J}\eta_1, \eta_3\big] \Big\rangle_{T_u\mathcal{X}} + \Big\langle \eta_3, D_u\mathbb{J}(u)\big[\mathbb{J}\eta_2, \eta_1\big] \Big\rangle_{T_u\mathcal{X}} = 0. \tag{2.11}$$

Proof. Given two Banach spaces X and Y. Let Y^* denote the dual space of Y and let $\big\langle \cdot, \cdot \big\rangle_Y$ denote the dual pairing in Y. The dual pairing can be seen as a product in the sense of [Růž04, Def. 2.6, p. 45]. In particular, this implies that for any Fréchet

[5] See [Mie11], [Mie13] and [Mie15] for more details.
[6] See for example [Mie11], [Mie13] and [MiT].

differentiable functions $f : X \longrightarrow Y^*$ and $g : X \longrightarrow Y$ the function

$$h : \begin{cases} X \longrightarrow \mathbb{R}, \\ x \longmapsto \big\langle f(x), g(x) \big\rangle_Y \end{cases} \tag{2.12}$$

is Fréchet differentiable. Moreover, due to the product rule in the version found in [Růž04, Th. 2.7, p. 45], we have

$$\forall\, \xi \in X : \qquad h'(x)[\xi] = \big\langle f'(x)[\xi],\, g(x) \big\rangle_Y + \big\langle f(x),\, g'(x)[\xi] \big\rangle_Y. \tag{2.13}$$

From the symmetry of the operator $\mathbb{J}$ we may infer the following for the double Poisson brackets for all sufficiently smooth functionals $\mathcal{F}_j,\, \mathcal{F}_k,\, \mathcal{F}_l : \mathcal{X} \longrightarrow \mathbb{R},\ j,\, k,\, l \in \{1, 2, 3\}$

$$\begin{aligned}
&\big\{ \{\mathcal{F}_j, \mathcal{F}_k\}_{\mathbb{J}}(u),\, \mathcal{F}_l \big\}_{\mathbb{J}}(u) = \Big\langle D_u \big\langle D_u\mathcal{F}_j(u),\, \mathbb{J} D_u \mathcal{F}_k(u) \big\rangle_{T_u\mathcal{X}},\, \mathbb{J} D_u \mathcal{F}_l(u) \Big\rangle_{T_u\mathcal{X}} \\
&= D_u \Big\langle D_u\mathcal{F}_j(u),\, \mathbb{J} D_u \mathcal{F}_k(u) \Big\rangle_{T_u\mathcal{X}} \big[\mathbb{J} D_u \mathcal{F}_l(u)\big] \\
&= \Big\langle D_u^2 \mathcal{F}_j(u) \big[\mathbb{J} D_u \mathcal{F}_l(u)\big],\, \mathbb{J} D_u \mathcal{F}_k(u) \Big\rangle_{T_u\mathcal{X}} + \Big\langle D_u\mathcal{F}_j(u),\, \mathbb{J} D_u^2 \mathcal{F}_k(u) \big[\mathbb{J} D_u \mathcal{F}_l(u)\big] \Big\rangle_{T_u\mathcal{X}} \\
&\quad + \Big\langle D_u\mathcal{F}_j(u),\, D_u\mathbb{J}(u) \big[\mathbb{J} D_u \mathcal{F}_l(u), D_u\mathcal{F}_k(u)\big] \Big\rangle_{T_u\mathcal{X}} \\
&= \Big\langle D_u^2 \mathcal{F}_j(u) \big[\mathbb{J} D_u \mathcal{F}_l(u)\big],\, \mathbb{J} D_u \mathcal{F}_k(u) \Big\rangle_{T_u\mathcal{X}} - \Big\langle D_u^2 \mathcal{F}_k(u) \big[\mathbb{J} D_u \mathcal{F}_l(u)\big],\, \mathbb{J} D_u \mathcal{F}_j(u) \Big\rangle_{T_u\mathcal{X}} \\
&\quad + \Big\langle D_u\mathcal{F}_j(u),\, D_u\mathbb{J}(u) \big[\mathbb{J} D_u \mathcal{F}_l(u), D_u\mathcal{F}_k(u)\big] \Big\rangle_{T_u\mathcal{X}}.
\end{aligned}$$

Denoting the derivatives of the functionals with $\eta_1,\, \eta_2,\, \eta_3$, correspondingly, and summing over $(j, k, l) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$, yields

$$\begin{aligned}
&\big\{ \{\mathcal{F}_1, \mathcal{F}_2\}_{\mathbb{J}}(u), \mathcal{F}_3 \big\}_{\mathbb{J}}(u) + \big\{ \{\mathcal{F}_2, \mathcal{F}_3\}_{\mathbb{J}}(u), \mathcal{F}_1 \big\}_{\mathbb{J}}(u) + \big\{ \{\mathcal{F}_3, \mathcal{F}_1\}_{\mathbb{J}}(u), \mathcal{F}_2 \big\}_{\mathbb{J}}(u) = \\
&\Big\langle \eta_1, D_u\mathbb{J}(u) \big[\mathbb{J}\eta_3, \eta_2\big] \Big\rangle_{T_u\mathcal{X}} + \Big\langle \eta_2, D_u\mathbb{J}(u) \big[\mathbb{J}\eta_1, \eta_3\big] \Big\rangle_{T_u\mathcal{X}} + \Big\langle \eta_3, D_u\mathbb{J}(u) \big[\mathbb{J}\eta_2, \eta_1\big] \Big\rangle_{T_u\mathcal{X}}
\end{aligned} \tag{2.14}$$

by using the symmetry of the mapping $(v, w) \longmapsto \langle D^2\mathcal{F}(u)[v], w\rangle$. If in the above setting the Poisson operator $\mathbb{J}$ is independent of $u \in \mathcal{X}$, the sum (2.14) is zero, thus, Jacobi's identity (2.5) is satisfied. Otherwise, Jacobi's identity (2.5) is satisfied if and only if (2.14) is zero, i. e. if and only if condition (2.11) is satisfied. This proves the lemma. □

Furthermore, the structure of the damped Hamiltonian system (2.3) is geometric in the sense that it is invariant under coordinate transformations. Considering another space $\mathcal{Y}$ that is isomorphic to $\mathcal{X}$ via a diffeomorphism

$$\tau : \begin{cases} \mathcal{Y} \longrightarrow \mathcal{X} \\ v \longmapsto u, \end{cases} \tag{2.15}$$

we define the transformed functional $\widehat{\mathcal{F}}_{\vartheta_*} : \mathcal{Y} \longrightarrow \mathbb{R}$ by

$$\widehat{\mathcal{F}}_{\vartheta_*}(v) := \mathcal{F}_{\vartheta_*}(\tau(v)). \tag{2.16}$$

The transformed Poisson and Onsager operator $\widehat{\mathbb{J}}(v)$, $\widehat{\mathbb{K}}(v)$ satisfy the properties (2.1)–(2.2) and are obtained via

$$\widehat{\mathbb{J}}(v) := D_u\tau^{-1}(u) \circ \mathbb{J}(u) \circ \big(D_u\tau^{-1}(u)\big)^* \tag{2.17a}$$

$$\widehat{\mathbb{K}}(v) := D_u\tau^{-1}(u) \circ \mathbb{K}(u) \circ \big(D_u\tau^{-1}(u)\big)^*. \tag{2.17b}$$

Here, for $\tau(v) = u$ we have $D_u\tau^{-1}(u) : T_v\mathcal{Y} \longrightarrow T_u\mathcal{X}$ and for its adjoint we have $(D_u\tau^{-1}(u))^* : T_v^*\mathcal{Y} \longrightarrow T^*_{u=\tau(v)}\mathcal{X}$. Thus, both $\widehat{\mathbb{J}}(v)$ and $\widehat{\mathbb{K}}(v)$ map the cotangent space $T_v^*\mathcal{Y}$ into the tangent space $T_v\mathcal{Y}$. The resulting transformed evolution equation in $T\mathcal{Y}$

$$\partial_t v = \Big(\widehat{\mathbb{J}}(v) - \frac{1}{\vartheta_*}\widehat{\mathbb{K}}(v)\Big) D\widehat{\mathcal{F}}_{\vartheta_*}(v) \tag{2.18}$$

is equivalent to the original evolution equation (2.3) in the sense that if v solves (2.18), then $u(t) = \tau(v(t))$ solves (2.3).

From the structures and symmetries of the Onsager and Poisson operators, we can formally state the following result.

Proposition 2.1.5. *Assume that* (2.3) *admits a solution* $u \in C^1([0,\infty),\mathcal{X})$ *subject to the initial condition* $u(0) = u_0 \in \mathcal{X}$. *Then, the following holds*

(i) If $\mathbb{K}(u) \equiv 0$, *then the initial free energy* $\mathcal{F}_{\vartheta_*}(u_0)$ *is a conserved quantity.*

(ii) In the case $\mathbb{K}(u) \geq 0$, *the free energy functional* $\mathcal{F}_{\vartheta_*}$ *is a Liapunov function for the solution* u *to the Cauchy problem for* (2.3), *i.e. we have*

$$\forall\, t \geq 0: \qquad \frac{d}{dt}\mathcal{F}_{\vartheta_*}(u(t)) = \big\langle D\mathcal{F}_{\vartheta_*}(u(t)), \partial_t u(t)\big\rangle_{T_{u(t)}\mathcal{X}} \leq 0. \tag{2.19}$$

(iii) In particular, we have the following estimate for all $t \geq 0$

$$\mathcal{F}_{\vartheta_*}(u(t)) \leq \mathcal{F}_{\vartheta_*}(u(t)) + \int_0^t \big\langle D\mathcal{F}_{\vartheta_*}(u(s)), \mathbb{K}(u(s))\, D\mathcal{F}_{\vartheta_*}(u(s))\big\rangle_{T_{u(s)}\mathcal{X}}\, ds = \mathcal{F}_{\vartheta_*}(u_0). \tag{2.20}$$

Proof. Due to (2.1) and (2.2) we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}_{\vartheta_*}(u(t)) &= \big\langle D\mathcal{F}_{\vartheta_*}(u(t)), \partial_t u(t)\big\rangle_{T_{u(t)}\mathcal{X}} \\
&= \big\langle D\mathcal{F}_{\vartheta_*}(u(t)), \big(\mathbb{J}(u(t)) - \mathbb{K}(u(t))\big) D\mathcal{F}_{\vartheta_*}(u(t))\big\rangle_{T_{u(t)}\mathcal{X}} \\
&= \big\langle D\mathcal{F}_{\vartheta_*}(u(t)), \mathbb{J}(u(t)) D\mathcal{F}_{\vartheta_*}(u(t))\big\rangle_{T_{u(t)}\mathcal{X}} - \big\langle D\mathcal{F}_{\vartheta_*}(u(t)), \mathbb{K}(u(t)) D\mathcal{F}_{\vartheta_*}(u(t))\big\rangle_{T_{u(t)}\mathcal{X}} \\
&= -\big\langle D\mathcal{F}_{\vartheta_*}(u(t)), \mathbb{K}(u(t)) D\mathcal{F}_{\vartheta_*}(u(t))\big\rangle_{T_{u(t)}\mathcal{X}} \leq 0.
\end{aligned} \tag{2.21}$$

Integrating over some time interval $(0, t)$ yields

$$\mathcal{F}_{\vartheta_*}(u(t)) - \mathcal{F}_{\vartheta_*}(u_0) = -\int_0^t \big\langle D\mathcal{F}_{\vartheta_*}(u(s)), \mathbb{K}(u(s))D\mathcal{F}_{\vartheta_*}(u(s))\big\rangle_{T_{u(s)}\mathcal{X}}\, ds. \tag{2.22}$$

In the case $\mathbb{K}(u) \equiv 0$, this implies $\mathcal{F}_{\vartheta_*}(u(t)) = \mathcal{F}_{\vartheta_*}(u_0)$ for all $t \geq 0$, thus $\mathcal{F}_{\vartheta_*}(u_0)$ is a conserved quantity. □

Remark 2.1.6. *Of course the calculations above remain formal, since for some concrete evolution quadrupel $(\mathcal{X}, \mathcal{F}_{\vartheta_*}, \mathbb{J}, \mathbb{K})$ the operators $\mathbb{J}$, $\mathbb{K}$ might only be densly defined on $T_u\mathcal{X}$. In such cases one has to make the arguments rigoruos.*

2.2. A Reduced Maxwell-Bloch Type Model

In Section 1.4 we derived the reduced unidirectional Maxwell-Bloch system (1.39) that describes the evolution of the complex *amplitude* a and the *inversion* w taking values in the interval $[-1, 1]$. If the direction of propagation is the x-axis, the resulting system is given by

$$\Big(\partial_x + \frac{1}{c_0}\partial_t\Big)a(x,t) = \frac{-i\mu_0 c_0}{\hbar}|\gamma_{12}|^2 a(x,t)\, w(x,t) \tag{2.23a}$$

$$\partial_t w(x,t) = -\frac{4T_2}{\hbar^2}|\gamma_{12}|^2|a(x,t)|^2 w(x,t) - \frac{1}{T_1}(w(x,t) - w_{\mathrm{eq}}(x)). \tag{2.23b}$$

In this system c_0, μ_0, $\hbar$, T_1, T_2 and $|\gamma_{12}|$ are positive constants.

Replacing a with ia and the term $w(x,t) - w_{\mathrm{eq}}(x)$ with $|w(x,t) - w_{\mathrm{eq}}(x)|$, the resulting system admits the form (2.3) with an operator $\mathbb{J}(a, w)$ and an Onsager operator $\mathbb{K}(a, w)$. The operator $\mathbb{J}(a, w)$ is the sum of two Poisson operators $\mathbb{J}_{\mathrm{ana}}(a, w)$ and $\mathbb{J}_{\mathrm{alg}}(a, w)$, however, the operator $\mathbb{J}(a, w)$ does not satisfy Jacobi's identity. In the case $\|w_{\mathrm{eq}}\|_{L^\infty(\Omega)} \leq 1$, the evolution takes place in the state space $\mathcal{X} = L^2(\Omega) \times \mathcal{W}$ where $\Omega \subsetneq \mathbb{R}$ is a given bounded open set and

$$\mathcal{W} := \Big\{ w \in L^1(\Omega) : \ \|w\|_{L^\infty(\Omega)} \leq 1 \Big\}. \tag{2.24}$$

The relevant topology of $\mathcal{W}$ is the one induced by the $L^1(\Omega)$-norm. In particular, we will only consider the dual space of $L^1(\Omega)$ and not the full dual space of $\mathcal{W}$. For simplicity we assume $\vartheta_* = 1$ for the rest of this section.

Next, we introduce the constants[7]

$$c_1 := \frac{c_0\mu_0\hbar}{4T_2} > 0, \qquad c_2 := \frac{c_0\mu_0}{\hbar}|\gamma_{12}|^2 > 0 \tag{2.25}$$

[7]One could also interpret these constants as functions depending on $x \in \Omega$ since this would be reasonable for ω_{21}, T_2 and γ_{12}, if different materials are present.

and define the free energy density f by

$$f(w) := \frac{c_0 \mu_0 \hbar}{4T_2} w = c_1 w. \tag{2.26}$$

In order to define an operator $\mathbb{J}(a,w)$ and an Onsager operator $\mathbb{K}(a,w)$ we introduce the functions γ and k in terms of the derivative of the free energy density f

$$\gamma(w) := -c_2 \frac{w}{f'(w)}, \tag{2.27}$$

$$k(w) := T_1^{-1} \frac{|w - w_{\mathrm{eq}}|}{f'(w)}. \tag{2.28}$$

Obviously, we have $k \geq 0$. We define the free energy functional $\mathcal{F}$ with its derivative $D\mathcal{F}$ by[8]

$$\mathcal{F}(a,w) = \int_\Omega \frac{1}{2}|a|^2 + f(w)\,dx, \qquad D\mathcal{F}(a,w) = (a, c_1) \tag{2.29}$$

and the operator $\mathbb{J}(a,w)$ and the Onsager operator $\mathbb{K}(a,w)$ by

$$\mathbb{J}(a,w) = \begin{pmatrix} -c_0\partial_x & -\gamma(w)\,a \\ \gamma(w)\,a & 0 \end{pmatrix}, \qquad \mathbb{K}(a,w) = \begin{pmatrix} 0 & 0 \\ 0 & k(w) \end{pmatrix}. \tag{2.30}$$

Recalling that we assumed $\vartheta_* = 1$, it is straightforward to see that with (2.29)–(2.30), the modified system (2.23) can in fact be written in the form (2.3).

Next, we generalize system (2.29)–(2.30). On the one hand, we keep the structure of the operators $\mathbb{J}(a,w)$, $\mathbb{K}(a,w)$ from (2.30) and the structure of the free energy functional $\mathcal{F}(a,w)$ from (2.29). On the other hand, we allow for a more general free energy density f and for more general functions γ, k in the operators.

Namely, we assume $\vartheta_* = 1 = c_0$ and that $\Omega \subsetneq \mathbb{R}$ is a bounded open set. For a given function $f \in C^2([-1,1],\mathbb{R})$, we consider the free energy functional[9] $\mathcal{F} : L^2(\Omega) \times \mathcal{W} \longrightarrow \mathbb{R}$ defined by

$$\mathcal{F}(a,w) = \int_\Omega \frac{1}{2}|a|^2 + f(w)\,dx. \tag{2.31}$$

Furthermore, for given functions $\gamma \in C^1([-1,1],\mathbb{R})$, $k \in C^0([-1,1],\mathbb{R})$ satisfying $k(w) \geq 0$ for all $w \in [-1,1]$ and $\gamma(\pm 1) = 0$, we consider the operators

$$\mathbb{J}(a,w) = \begin{pmatrix} -\partial_x & -\gamma(w)\,a \\ \gamma(w)\,a & 0 \end{pmatrix}, \qquad \mathbb{K}(a,w) = \begin{pmatrix} 0 & 0 \\ 0 & k(w) \end{pmatrix}. \tag{2.32}$$

For the above operators, the next proposition holds.

[8] In fact, we interpret the free energy functional as a pure energy functional. However, we denote it with $\mathcal{F}$ and not with $\mathcal{E}$.

[9] We still interpret the free energy functional as a pure energy functional, but denote it with $\mathcal{F}$.

Proposition 2.2.1. *The operator $\mathbb{J}(a,w)$ from* (2.32) *is the sum of two Poisson operators. These Poisson operators and the Onsager operator from* (2.32) *satisfy the conditions of Definition 2.1.1. This means*

(i) There exist operators $\mathbb{J}_{\mathrm{ana}}(a,w)$, $\mathbb{J}_{\mathrm{alg}}(a,w) : \big(H_0^1(\Omega)\times L^1(\Omega)\big)^* \longrightarrow L^2(\Omega)\times L^1(\Omega)$ such that $\mathbb{J}(a,w) = \mathbb{J}_{\mathrm{ana}}(a,w) + \mathbb{J}_{\mathrm{alg}}(a,w)$. These operators are skew symmetric and satisfy Jacobi's identity.

(ii) The operator $\mathbb{K}(a,w) : \big(H_0^1(\Omega)\times L^1(\Omega)\big)^* \longrightarrow L^2(\Omega)\times L^1(\Omega)$ is symmetric and positive semi-definite.

Proof. (i) Clearly, the operator $\mathbb{J}(a,w)$ is the sum of the operators

$$\mathbb{J}_{\mathrm{ana}}(a,w) = \begin{pmatrix} -\partial_x & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbb{J}_{\mathrm{alg}}(a,w) = \begin{pmatrix} 0 & -\gamma(w)\,a \\ \gamma(w)\,a & 0 \end{pmatrix}. \tag{2.33}$$

It is clear that the statements hold for $\mathbb{J}_{\mathrm{ana}}$ due to Lemma 2.1.4. In order to show the symmetry condition and Jacobi's identity for $\mathbb{J}_{\mathrm{alg}}$ we identify $\big(H_0^1(\Omega)\times L^1(\Omega)\big)^*$ with $H_0^1(\Omega)\times L^\infty(\Omega)$. For all $(a,w)\in H_0^1(\Omega)\times L^1(\Omega)$ and for all $(\alpha_j,\omega_j)\in H_0^1(\Omega)\times L^\infty(\Omega)$, $j\in\{1,2\}$ we have

$$\begin{aligned}
&\int_\Omega \begin{pmatrix}\alpha_1\\ \omega_1\end{pmatrix}\cdot \mathbb{J}_{\mathrm{alg}}(a,w)\left[\begin{pmatrix}\alpha_2\\ \omega_2\end{pmatrix}\right] dx = \int_\Omega \begin{pmatrix}\alpha_1\\ \omega_1\end{pmatrix}\cdot\begin{pmatrix}-\gamma(w)\,a\,\omega_2\\ \gamma(w)\,a\,\alpha_2\end{pmatrix} dx\\
&= \int_\Omega -\alpha_1\gamma(w)\,a\,\omega_2 + \omega_1\gamma(w)\,a\,\alpha_2\,dx = -\int_\Omega \big(-\alpha_2\gamma(w)\,a\,\omega_1 + \omega_2\gamma(w)\,a\,\alpha_1\big)\,dx\\
&= \int_\Omega \begin{pmatrix}\alpha_2\\ \omega_2\end{pmatrix}\cdot\begin{pmatrix}\gamma(w)\,a\,\omega_1\\ -\gamma(w)\,a\,\alpha_1\end{pmatrix} dx = -\int_\Omega \begin{pmatrix}\alpha_2\\ \omega_2\end{pmatrix}\cdot \mathbb{J}_{\mathrm{alg}}(a,w)\left[\begin{pmatrix}\alpha_1\\ \omega_1\end{pmatrix}\right] dx.
\end{aligned}\tag{2.34}$$

This shows the symmetry condition (2.1).

Next, we show Jacobi's identity for $\mathbb{J}_{\mathrm{alg}}$. For brevity, we write $\mathbb{J}$ instead of $\mathbb{J}_{\mathrm{alg}}$ in this proof. For given functionals $\mathcal{F}_j : L^2(\Omega)\times L^1(\Omega)\longrightarrow\mathbb{R}$, $j\in\{1,2,3\}$ let $(\alpha_j,\omega_j)\in H_0^1(\Omega)\times L^\infty(\Omega)$, $j\in\{1,2,3\}$ denote the first order derivatives of $\mathcal{F}_j$ w.r.t. (a,w). Then, for all $(a,w)\in H_0^1(\Omega)\times L^1(\Omega)$ we have for $j,\,k,\,l\in\{1,2,3\}$

$$\begin{aligned}
&\big\{\{\mathcal{F}_j,\mathcal{F}_k\}_{\mathbb{J}}(a,w),\mathcal{F}_l\big\}_{\mathbb{J}}(a,w) = \int_\Omega D_{(a,w)}\left(\begin{pmatrix}\alpha_j\\ \omega_j\end{pmatrix}\cdot\mathbb{J}(a,w)\left[\begin{pmatrix}\alpha_k\\ \omega_k\end{pmatrix}\right]\right)\cdot\mathbb{J}(a,w)\begin{pmatrix}\alpha_l\\ \omega_l\end{pmatrix} dx\\
&= \int_\Omega \begin{pmatrix} D_a\big(\gamma(w)\,a(\alpha_k\omega_j-\alpha_j\omega_k)\big)\\ D_w\big(\gamma(w)\,a(\alpha_k\omega_j-\alpha_j\omega_k)\big)\end{pmatrix}\cdot\begin{pmatrix}-\gamma(w)\,a\,\omega_l\\ \gamma(w)\,a\,\alpha_l\end{pmatrix} dx\\
&= \int_\Omega |a|^2\gamma(w)\,\alpha_l\big(\gamma'(w)(\alpha_k\omega_j-\alpha_j\omega_k) + \gamma(w)D_w(\alpha_k\omega_j-\alpha_j\omega_k)\big)\\
&\qquad - |\gamma(w)|^2 a\,\omega_l\big((\alpha_k\omega_j-\alpha_j\omega_k) + aD_a(\alpha_k\omega_j-\alpha_j\omega_k)\big)\,dx.
\end{aligned}\tag{2.35}$$

Clearly, by summing over $(j,k,l)\in\{(1,2,3),(2,3,1),(3,1,2)\} =: I$ we get the following identity from the symmetry $D_a\omega_j = D_w\alpha_j$ for all $j\in\{1,2,3\}$

$$\begin{aligned}
&\sum_{(j,k,l)\in I} \big\{\{\mathcal{F}_j, \mathcal{F}_k\}_{\mathbb{J}}(a,w), \mathcal{F}_l\big\}_{\mathbb{J}}(a,w) \\
&= |a|^2|\gamma(w)|^2 \sum_{(j,k,l)\in I} \alpha_l D_w(\alpha_k\omega_j - \alpha_j\omega_k) - \omega_l D_a(\alpha_k\omega_j - \alpha_j\omega_k) = 0.
\end{aligned} \tag{2.36}$$

This proves Jacobi's identity.
(ii) Due to the structure of $\mathbb{K}(a,w)$ and $k \geq 0$ the statements are obvious. □

Remark 2.2.2. *The operator* $\mathbb{J}(a,w)$ *from* (2.32) *is skew symmetric but does not satisfy Jacobi's identity.*

Proof. We set $u := (a,w)$. Clearly, $\mathbb{J}(a,w)$ is skew symmetric. In the following, $\langle \cdot, \cdot \rangle$ denotes the dual pairing in $H_0^1(\Omega) \times L^1(\Omega)$. For arbitrary $\eta_j := (\alpha_j, \omega_j) \in H_0^1(\Omega) \times L^\infty(\Omega)$, $j \in \{1,2,3\}$, by using the product rule, the skew symmetry of $\mathbb{J}(u)$ and the symmetry of $D_u\eta_j$, we get for the Poisson bracket

$$\begin{aligned}
&\Big\langle D_u\big\langle \eta_j,\, \mathbb{J}(u)\eta_k\big\rangle,\, \mathbb{J}(u)\eta_l\Big\rangle = D_u\big\langle \eta_j,\, \mathbb{J}(u)\eta_k\big\rangle\big[\mathbb{J}(u)\eta_l\big] \\
&= \Big\langle D_u\eta_j\big[\mathbb{J}(u)\eta_l\big],\, \mathbb{J}(u)\eta_k\Big\rangle + \Big\langle \eta_j,\, \big(\mathbb{J}(u)\circ D_u\eta_k\big)\big[\mathbb{J}(u)\eta_l\big]\Big\rangle \\
&\quad + \left\langle \begin{pmatrix}\alpha_j\\ \omega_j\end{pmatrix}, \begin{pmatrix} -\gamma(w)\omega_k & -\gamma'(w)a\omega_k \\ \gamma(w)\alpha_k & \gamma'(w)a\alpha_k \end{pmatrix}\left[\begin{pmatrix} -\partial_x\alpha_l - \gamma(w)a\omega_l \\ \gamma(w)a\alpha_l \end{pmatrix}\right]\right\rangle \\
&= \Big\langle D_u\eta_j\big[\mathbb{J}(u)\eta_l\big],\, \mathbb{J}(u)\eta_k\Big\rangle - \Big\langle D_u\eta_k\big[\mathbb{J}(u)\eta_l\big],\, \mathbb{J}(u)\eta_j\Big\rangle \\
&\quad + \left\langle \begin{pmatrix}\alpha_j\\ \omega_j\end{pmatrix}, \begin{pmatrix} -\gamma(w)\omega_k & -\gamma'(w)a\omega_k \\ \gamma(w)\alpha_k & \gamma'(w)a\alpha_k \end{pmatrix}\left[\begin{pmatrix} -\partial_x\alpha_l - \gamma(w)a\omega_l \\ \gamma(w)a\alpha_l \end{pmatrix}\right]\right\rangle
\end{aligned} \tag{2.37}$$

In the sum over $(j,k,l) \in I := \{(1,2,3), (2,3,1), (3,1,2)\}$ the first two summands are equal to zero. The third summand is equal to $\Big\langle \eta_j,\, D_u\mathbb{J}(u)\big[\eta_k,\, \mathbb{J}(u)\eta_l\big]\Big\rangle$ and we have

$$\begin{aligned}
&\Big\langle \eta_j,\, D_u\mathbb{J}(u)\big[\eta_k,\, \mathbb{J}(u)\eta_l\big]\Big\rangle \\
&= \left\langle \begin{pmatrix}\alpha_j\\ \omega_j\end{pmatrix}, \begin{pmatrix} \big(\gamma(w)\omega_k\partial_x\alpha_l + |\gamma(w)|^2 a\omega_k\omega_l\big) - \gamma'(w)\gamma(w)|a|^2\omega_k\alpha_l \\ \big(-\gamma(w)\alpha_k\partial_x\alpha_l - |\gamma(w)|^2 a\alpha_k\omega_l\big) + \gamma'(w)\gamma(w)|a|^2\alpha_l\alpha_k \end{pmatrix}\right\rangle \\
&= \int_\Omega \Big(\big(\gamma(w)\omega_k\alpha_j\partial_x\alpha_l + |\gamma(w)|^2 a\alpha_j\omega_k\omega_l\big) - \gamma'(w)\gamma(w)|a|^2\omega_k\alpha_l\alpha_j \\
&\qquad - \big(\gamma(w)\omega_j\alpha_k\partial_x\alpha_l + |\gamma(w)|^2 a\alpha_k\omega_l\omega_j\big) + \gamma'(w)\gamma(w)|a|^2\omega_j\alpha_k\alpha_l\Big)\, dx \\
&= \int_\Omega \Big(|\gamma(w)|^2 a\big(\alpha_j\omega_k\omega_l - \alpha_k\omega_l\omega_j\big) + \gamma'(w)\gamma(w)|a|^2\big(\omega_j\alpha_k\alpha_l - \omega_k\alpha_l\alpha_j\big) && (2.38)\\
&\qquad + \gamma(w)\big(\omega_k\partial_x\alpha_l\alpha_j - \omega_j\alpha_k\partial_x\alpha_l\big)\Big)\, dx. && (2.39)
\end{aligned}$$

Summing over $(j,k,l) \in I$ the terms in (2.38) are equal to zero, but for (2.39) we get

$$\sum_{(j,k,l)\in I} \int_\Omega \gamma(w)\big(\omega_k\partial_x\alpha_l\alpha_j - \omega_j\alpha_k\partial_x\alpha_l\big)\, dx = \int_\Omega \gamma(w)\, \vec{\omega}\cdot\big(\partial_x\vec{\alpha}\times\vec{\alpha}\big)\, dx \tag{2.40}$$

with $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$. In general it does not hold $\big(\partial_x\vec{\alpha}\times\vec{\alpha}\big) = 0$. □

Inserting (2.31)–(2.32) into the general evolution equation for damped Hamiltonian system (2.3) with $\vartheta_* = 1$ yields the following system of partial differential equations

$$\partial_t a = -\partial_x a - \gamma(w) f'(w) a \tag{2.41a}$$
$$\partial_t w = \gamma(w)|a|^2 - k(w) f'(w). \tag{2.41b}$$

Next, we will give a concrete example for functions f, γ and k. We choose the free energy density f to be the logarithmic potential

$$f(w) = (1-w)\log(1-w) + (1+w)\log(1+w) \tag{2.42}$$

and define the functions γ and k in terms of the derivative of the function f by

$$\gamma(w) = \alpha \frac{w}{f'(w)}, \qquad k(w) = \kappa \frac{w}{f'(w)} \tag{2.43}$$

with constants $\alpha \in \mathbb{R}$ and $\kappa \geq 0$. The functions f, γ, k are defined for $w \in (-1,1)$. We note that the logarithmic potential f satisfies the following asymptotics

$$\lim_{w\to -1} f(w) = 2\log 2, \qquad \lim_{w\to +1} f(w) = 2\log 2, \qquad \text{as well as } f(0) = 0. \tag{2.44}$$

We continuously extend f to $[-1,1]$ and for $|w| > 1$ we set $f(w) = \infty$. The derivative of the function f is given by

$$f'(w) = \log\left(\frac{1+w}{1-w}\right) = \log(1+w) - \log(1-w) \tag{2.45}$$

and satisfies

$$\lim_{w\to -1} f'(w) = -\infty, \qquad \lim_{w\to +1} f'(w) = +\infty. \tag{2.46}$$

We extend f' to $[-1,1]$ by setting

$$f'(w) := \begin{cases} -\infty, & w = -1 \\ \log(1+w) - \log(1-w), & w \in (-1,1) \\ +\infty, & w = 1. \end{cases} \tag{2.47}$$

For notational convenience we define the function $h : \mathbb{R} \longrightarrow [0, 1/2]$ by[10]

$$h(w) := \frac{w}{f'(w)} = \frac{w}{\log(1+w) - \log(1-w)} \tag{2.48}$$

and note that h satisfies $h(w) = 0$ if $|w| = 1$. For $|w| \geq 1$ we set $h(w) = 0$. Then, for the above choices, the operator $\mathbb{J}(a,w)$ and the Onsager operator $\mathbb{K}(a,w)$ are given by

$$\mathbb{J}(a,w) = \begin{pmatrix} -\partial_x & -\alpha h(w)\, a \\ \alpha h(w)\, a & 0 \end{pmatrix}, \qquad \mathbb{K}(a,w) = \begin{pmatrix} 0 & 0 \\ 0 & \kappa h(w) \end{pmatrix}. \tag{2.49}$$

[10] A plot of the function h can be found in Figure 3.1 on page 43.

Moreover, for the above choice, the derivative of the free energy is given by[11]

$$D\mathcal{F}(a,w) = \big(a,\, \log(1+w) - \log(1-w)\big). \tag{2.50}$$

Inserting (2.49)–(2.50) into the general evolution equation for damped Hamiltonian systems (2.3) with $\vartheta_* = 1$ yields the following system for the unknown functions (a, w)

$$\partial_t a(x,t) = -\partial_x a(x,t) - \alpha\, w(x,t)\, a(x,t) \qquad \text{in } \Omega \tag{2.51a}$$
$$\partial_t w(x,t) = \alpha\, h(w)\, |a(x,t)|^2 - \kappa\, w(x,t) \qquad \text{in } \Omega. \tag{2.51b}$$

We will perform a thorough mathematical study of this system in Chapter 3.

2.3. Maxwell's Equations

A simple example of an undamped Hamiltonian system in the state space $\mathcal{X} = L^2(\Omega;\mathbb{R}^3_{\mathbf{D}}) \times L^2(\Omega;\mathbb{R}^3_{\mathbf{B}})$ with $\Omega \subseteq \mathbb{R}^3$ is given by the time dependent Maxwell equations in the coordinates $\big(\mathbf{D},\mathbf{B}\big) \in L^2(\Omega;\mathbb{R}^3_{\mathbf{D}}) \times L^2(\Omega;\mathbb{R}^3_{\mathbf{B}})$ in a regime without free currents. The field $\mathbf{D}$ is called the *electric displacement field*. We call the field $\mathbf{B}$ the *magnetic induction*.[12]

Our considered Hamiltonian system consists of the energy functional

$$\mathcal{E}\big(\mathbf{D},\mathbf{B}\big) = \int_\Omega \frac{1}{2\epsilon_0}\big|\mathbf{D}-\mathbf{P}\big|^2 + \frac{1}{2\mu_0}\big|\mathbf{B}-\mathbf{M}\big|^2\, dx \tag{2.52}$$

with the *polarization (field)* $\mathbf{P} \in L^2(\Omega;\mathbb{R}^3_{\mathbf{D}})$ and the *magnetization* $\mathbf{M} \in L^2(\Omega;\mathbb{R}^3_{\mathbf{B}})$, as well the Poisson operator[13]

$$\mathbb{J}\big(\mathbf{D},\mathbf{B}\big) = \begin{pmatrix} 0 & \operatorname{curl} \\ -\operatorname{curl} & 0 \end{pmatrix}. \tag{2.53}$$

Assuming that both $\mathbf{P}$ and $\mathbf{M}$ do not depend on $\big(\mathbf{D},\mathbf{B}\big)$, the derivative $D\mathcal{E}\big(\mathbf{D},\mathbf{B}\big) \in \big(L^2(\Omega;\mathbb{R}^3_{\mathbf{D}}) \times L^2(\Omega;\mathbb{R}^3_{\mathbf{B}})\big)^*$ of the energy functional $\mathcal{E}$ is given by

$$D\mathcal{E}\big(\mathbf{D},\mathbf{B}\big) = \big(\epsilon_0^{-1}(\mathbf{D}-\mathbf{P}),\, \mu_0^{-1}(\mathbf{B}-\mathbf{M})\big). \tag{2.54}$$

With the *electric field* $\mathbf{E} := \epsilon_0^{-1}\big(\mathbf{D}-\mathbf{P}\big) \in L^2(\Omega;\mathbb{R}^3_{\mathbf{E}})$, the above undamped Hamiltonian system yields the time dependent Maxwell equations in a regime without free currents[14]

[11] We note that the function $\log(1+w) - \log(1-w) \in L^\infty(\Omega)$ is well defined if $\|w\|_{L^\infty(\Omega)} < 1$.

[12] We call the field $\mathbf{H} := \mu_0^{-1}\mathbf{B} - \mathbf{M}$ the *magnetic field*. In order to distinguish between the two fields we do not call $\mathbf{B}$ magnetic field in contrast to many others.

[13] Clearly, the operator $\mathbb{J}(\mathbf{D},\mathbf{B})$ is only densely defined on $L^2(\Omega;\mathbb{R}^3_{\mathbf{D}}) \times L^2(\Omega;\mathbb{R}^3_{\mathbf{B}})$.

[14] The standard reference from a physical point of view is [Jac06] (or rather the original version of Wiley from 1975). In [Rau12, p. 44] a case with an external current is studied. See also [Tay96].

$$\partial_t \mathbf{D} = +\mu_0^{-1} \operatorname{curl} \big(\mathbf{B} - \mathbf{M}\big) = \operatorname{curl} \mathbf{H} \qquad \text{in } \Omega \tag{2.55a}$$
$$\partial_t \mathbf{B} = -\epsilon_0^{-1} \operatorname{curl} \big(\mathbf{D} - \mathbf{P}\big) = -\operatorname{curl} \mathbf{E} \qquad \text{in } \Omega. \tag{2.55b}$$

The full set of Maxwell equations in a regime without free currents and free charges is achieved by adding the time independent Maxwell equations

$$\operatorname{div} \mathbf{D} = 0 \qquad \text{and} \qquad \operatorname{div} \mathbf{B} = 0 \qquad \text{in } \Omega. \tag{2.56}$$

We end this section by showing that for sufficiently smooth fields, the operator $\mathbb{J}(\mathbf{D}, \mathbf{B})$ satisfies the symmetry condition (2.1). Since $\mathbb{J} = \mathbb{J}(\mathbf{D}, \mathbf{B})$ is independent of $(\mathbf{D}, \mathbf{B})$, Lemma 2.1.4 implies that then the structural property (2.2a) is also satisfied. In the following, we identify $\big(H_0^1(\Omega; \mathbb{R}^3_{\mathbf{D}}) \times H_0^1(\Omega; \mathbb{R}^3_{\mathbf{B}})\big)^*$ with $H_0^1(\Omega; \mathbb{R}^3_{\mathbf{D}}) \times H_0^1(\Omega; \mathbb{R}^3_{\mathbf{B}})$ and interpret $\mathbb{J}$ as an operator mapping $H_0^1(\Omega; \mathbb{R}^3_{\mathbf{D}}) \times H_0^1(\Omega; \mathbb{R}^3_{\mathbf{B}})$ into $L^2(\Omega; \mathbb{R}^3_{\mathbf{D}}) \times L^2(\Omega; \mathbb{R}^3_{\mathbf{B}})$.

Proposition 2.3.1. *The operator* $\mathbb{J} : H_0^1(\Omega; \mathbb{R}^3_{\mathbf{D}}) \times H_0^1(\Omega; \mathbb{R}^3_{\mathbf{B}}) \longrightarrow L^2(\Omega; \mathbb{R}^3_{\mathbf{D}}) \times L^2(\Omega; \mathbb{R}^3_{\mathbf{B}})$ *satisfies the conditions of Definition 2.1.1.*

Proof. In view of Lemma 2.1.4 it suffices to show that for all $\big(\mathbf{D}_j, \mathbf{B}_j\big) \in H_0^1(\Omega; \mathbb{R}^3_{\mathbf{D}}) \times H_0^1(\Omega; \mathbb{R}^3_{\mathbf{B}})$, $j \in \{1, 2\}$ it holds

$$\Big\langle \big(\mathbf{D}_1, \mathbf{B}_1\big), \, \mathbb{J}\big[\big(\mathbf{D}_2, \mathbf{B}_2\big)\big] \Big\rangle_{L^2(\Omega)} = -\Big\langle \mathbb{J}\big[\big(\mathbf{D}_1, \mathbf{B}_1\big)\big], \, \big(\mathbf{D}_2, \mathbf{B}_2\big) \Big\rangle_{L^2(\Omega)}. \tag{2.57}$$

As shown in Section A.5 we can write the operator $\mathbb{J}$ as the sum

$$\mathbb{J} = \begin{pmatrix} 0 & \operatorname{curl} \\ -\operatorname{curl} & 0 \end{pmatrix} = -\sum_{j=1}^{3} A_j \partial_{x_j} \tag{2.58}$$

with A_1, A_2, A_3 given by (A.144). Due to the symmetry of A_1, A_2, A_3 we have for all $\big(\mathbf{D}_j, \mathbf{B}_j\big) \in H_0^1(\Omega; \mathbb{R}^3_{\mathbf{D}}) \times H_0^1(\Omega; \mathbb{R}^3_{\mathbf{B}})$, $j \in \{1, 2\}$ the equality

$$\begin{aligned}
&\Big\langle \big(\mathbf{D}_1, \mathbf{B}_1\big), \, \mathbb{J}\big[\big(\mathbf{D}_2, \mathbf{B}_2\big)\big] \Big\rangle_{L^2(\Omega)} \\
&= -\int_\Omega \big(\mathbf{D}_1, \mathbf{B}_1\big) \cdot \sum_{j=1}^{3} A_j \partial_{x_j} \big(\mathbf{D}_2, \mathbf{B}_2\big) \, dx = \int_\Omega \sum_{j=1}^{3} A_j^T \partial_{x_j} \big(\mathbf{D}_1, \mathbf{B}_1\big) \cdot \big(\mathbf{D}_2, \mathbf{B}_2\big) \, dx \\
&= \int_\Omega \sum_{j=1}^{3} A_j \partial_{x_j} \big(\mathbf{D}_1, \mathbf{B}_1\big) \cdot \big(\mathbf{D}_2, \mathbf{B}_2\big) \, dx = -\Big\langle \mathbb{J}\big[\big(\mathbf{D}_1, \mathbf{B}_1\big)\big], \, \big(\mathbf{D}_2, \mathbf{B}_2\big) \Big\rangle_{L^2(\Omega)}
\end{aligned}$$

by partial integration. This proves (2.57). □

2.4. A Dissipative Quantum Model

In [Mie13] following [Ött10], [Ött11] a GENERIC system has been proposed that describes the evolution of a density matrix for an N-dimensional Hilbert space in the state space $L^1(\Omega;\mathbb{C}^{N\times N}_{\text{herm}})$ where[15]

$$\Omega \subsetneq \mathbb{R}^3 \quad \text{is a bounded open set.} \tag{2.59}$$

In fact, there is a need to impose more properties on the state space, so that it correctly pictures the admissible set for density matrices. This leads to the space

$$\mathcal{R}_N := \Big\{\rho \in (L^1 \cap L^\infty)(\Omega;\mathbb{C}^{N\times N}_{\text{herm}}) : \forall_{\text{a.a.}}\ x \in \Omega : \rho(x) \geq 0,\ \mathrm{Tr}\rho(x) = 1\Big\} \tag{2.60}$$

where $\rho \geq 0$ means that ρ is positive semi-definite. Nevertheless, the relevant topology is the one induced by the $L^1(\Omega;\mathbb{C}^{N\times N}_{\text{herm}})$-norm. In particular, we will only consider the dual space of $L^1(\Omega;\mathbb{C}^{N\times N}_{\text{herm}})$ and not the full dual space of $\mathcal{R}_N$. However, the GENERIC system considered in [Mie13] involves the canonical correlation operator $\mathcal{C}_\rho$ defined by

$$\mathcal{C}_\rho : \begin{cases} L^\infty(\Omega;\mathbb{C}^{N\times N}_{\text{herm}}) \longrightarrow (L^1 \cap L^\infty)(\Omega;\mathbb{C}^{N\times N}_{\text{herm}}) \\ \qquad\qquad A \longmapsto \int_0^1 \rho^s A \rho^{1-s}\, ds. \end{cases} \tag{2.61}$$

The canonical correlation operator $\mathcal{C}_\rho$ satisfies[16] the following "miracle relation" for all $A \in L^\infty(\Omega;\mathbb{C}^{N\times N}_{\text{herm}})$ and for all $\rho \in \mathcal{R}_N$ (for a proof see [Mie13])

$$\big[\mathcal{C}_\rho A,\, \log\rho\big] = \big[A,\, \rho\big] = \mathcal{C}_\rho\big[A,\, \log\rho\big]. \tag{2.62}$$

Starting from this GENERIC system, in [Mie13] also a damped Hamiltonian system was derived by means of an adiabatic limit procedure. We continue the study of this damped Hamiltonian system. The considered isothermal free energy functional is given by

$$\mathcal{F}_{\vartheta_*}(\rho) = \int_\Omega \mathrm{Tr}\big(\rho\mathsf{H}\big) + \vartheta_* k_B \mathrm{Tr}\big(\rho\log\rho\big)\, dx \tag{2.63}$$

with the Bloch-system Hamiltonian $\mathsf{H} \in L^\infty(\Omega;\mathbb{C}^{N\times N}_{\text{herm}})$, a fixed temperature $\vartheta_* > 0$ and the Boltzmann constant $k_B > 0$. The Poisson operator is given by

$$\mathcal{J}(\rho) : \begin{cases} \big(L^1(\Omega;\mathbb{C}^{N\times N}_{\text{herm}})\big)^* \longrightarrow L^1(\Omega;\mathbb{C}^{N\times N}_{\text{herm}}), \\ \qquad\qquad\qquad \xi \longmapsto \frac{i}{\hbar}\big[\rho,\, \xi\big] \end{cases} \tag{2.64}$$

[15]We consider a special case of the situation from [Mie13].
[16]An explanation what is meant by $\log\rho$ is given below.

and the Onsager operator is given by

$$\mathcal{K}(\rho):\begin{cases}\big(L^1(\Omega;\mathbb{C}^{N\times N}_{\text{herm}})\big)^* \longrightarrow L^1(\Omega;\mathbb{C}^{N\times N}_{\text{herm}}),\\ \qquad\qquad\qquad \xi \longmapsto \sum_{m=1}^{M}\Big[Q_m,\mathcal{C}_\rho\big[Q_m,\,\xi\big]\Big]\end{cases} \tag{2.65}$$

with a finite number M of coupling operators $Q_m \in L^\infty(\Omega;\mathbb{C}^{N\times N}_{\text{herm}})$. The isothermal free energy functional is the sum $\mathcal{F}_{\vartheta_*} = \mathcal{E} - \vartheta_*\mathcal{S}$ consisting of an energy functional $\mathcal{E}$ and an entropy functional $\mathcal{S}$ given by

$$\mathcal{E}(\rho) = \int_\Omega \mathrm{Tr}\big(\rho\mathsf{H}\big)\,dx, \qquad \mathcal{S}(\rho) = -\int_\Omega k_B \mathrm{Tr}\big(\rho\log\rho\big)\,dx. \tag{2.66}$$

The expression $\log\rho$ is to be understood in the following sense. For every Hermitian matrix $H \in \mathbb{C}^{N\times N}_{\text{herm}}$ there exists a unitary matrix U such that $U^*HU = \mathrm{diag}(\lambda_1,\dots,\lambda_N)$, where $\lambda_1,\dots,\lambda_N$ are the eigenvalues of H. For the diagonal matrix $D := \mathrm{diag}(\lambda_1,\dots,\lambda_N)$ we have $\log D := \mathrm{diag}(\log\lambda_1,\dots,\log\lambda_N)$. For an arbitrary Hermitian matrix $H \in \mathbb{C}^{N\times N}_{\text{herm}}$ with spectral decomposition D, we define $\log\rho := U(\log D)U^*$. Moreover, using the inverse function theorem and the product rule[17], we get that for all $\rho \in \mathcal{R}_N$ we have

$$D_\rho\mathrm{Tr}\big(\rho\log\rho\big) = \log\rho + \mathsf{Id}_{N\times N}. \tag{2.67}$$

Since the term $D_\rho\mathrm{Tr}\big(\rho\log\rho\big)$ is commuted with some other matrix at all times and since commutation with $\mathsf{Id}_{N\times N}$ gives zero, we ignore the appearance of $\mathsf{Id}_{N\times N}$ and write

$$D_\rho\mathcal{F}_{\vartheta_*}(\rho) = \mathsf{H} + \vartheta_* k_B \log\rho. \tag{2.68}$$

Insertion of (2.64), (2.65) and (2.68) into the general evolution equation for damped Hamiltonian systems (2.3) leads to the following evolution equation for the density matrix ρ.

$$\partial_t\rho = \frac{i}{\hbar}\big[\rho,\mathsf{H}\big] - \frac{1}{\vartheta_*}\sum_{m=1}^{M}\Big[Q_m,\,\mathcal{C}_\rho\big[Q_m,\,\big(\mathsf{H} + k_B\vartheta_*\log\rho\big)\big]\Big] \qquad \text{in } \Omega. \tag{2.69}$$

We stress that for all $A,\,B \in \mathbb{C}^{N\times N}_{\text{herm}}$ with arbitrary $N \in \mathbb{N}$ it holds $\mathrm{Tr}\big[A,B\big] = 0$ (see Lemma A.1.19). Therefore, the trace of ρ is left invariant by the evolution equation (2.69). In the next subsection, we introduce Bloch coordinates for the simplest two-level case. Since our main interest lies in analyzing this simple case, we will only prove that the Poisson and Onsager operators satisfy (2.1)–(2.2) for this simple case. This will be done at the end of the next subsection.

[17] See for example [Růž04, p. 45–54] and note that for martrices $A,\,B \in \mathbb{C}^{N\times N}_{\text{herm}}$ the expression $\mathrm{Tr}\big(AB\big)$ is a product in the sense of [Růž04, Def. 2.6, p. 45].

2.4.1. Evolution in Bloch coordinates

In Section A.2 Bloch coordinates $(a_0, \boldsymbol{a}) \in \mathbb{R}^4 \cong \mathbb{C}^{2\times 2}_{\text{herm}}$ are introduced for the two-level case via the diffeomorphism $\tau^{-1} : \mathbb{C}^{2\times 2}_{\text{herm}} \longrightarrow \mathbb{R}^4$ mapping the matrix $A \in \mathbb{C}^{2\times 2}_{\text{herm}}$ with the entries $\{a_{jk}\}_{j,k=1,2}$ to the vector $(a_0, \boldsymbol{a})^T$ according to

$$\tau^{-1} : \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longmapsto \begin{pmatrix} (a_{11}+a_{22})/2 \\ (a_{11}-a_{22})/2 \\ (a_{12}+a_{21})/2 \\ (a_{12}-a_{21})/(2i) \end{pmatrix} \tag{2.70}$$

by setting $(a_0, \boldsymbol{a})^T := \tau^{-1}(A)$. The matrix A, on the other hand, can be recovered from its Bloch coordinates via the mapping $\tau : \mathbb{R}^4 \longrightarrow \mathbb{C}^{2\times 2}_{\text{herm}}$ (see Chapter A.2 for details). In particular, we identify the set $\mathcal{R}_2$ with the set

$$\mathcal{A} := \Big\{ \boldsymbol{a} = (a_1, a_2, a_3) \in (L^1 \cap L^\infty)(\Omega; \mathbb{R}^3_{\boldsymbol{a}}) : \ \|\boldsymbol{a}\|_{L^\infty(\Omega)} \le 1/2 \Big\} \tag{2.71}$$

via

$$\mathcal{R}_2 \ni \rho = \widehat{\rho}(\boldsymbol{a}) := \tau(1/2, \boldsymbol{a}) = \begin{pmatrix} \frac{1}{2} + a_1 & a_2 + ia_3 \\ a_2 - ia_3 & \frac{1}{2} - a_1 \end{pmatrix} \tag{2.72}$$

and call $\boldsymbol{a}$ the Bloch vector of the density matrix ρ. We note that the evolution of the density matrix ρ given by (2.69) actually takes place in the set $\mathcal{R}_2$ since the trace is left invariant by the evolution equation. Therefore, we would expect that the evolution of the Bloch coordinates $(a_0, \boldsymbol{a})$ has a non-zero contribution only in the component $\boldsymbol{a}$.

In the following, we transform the damped Hamiltonian system (2.63)–(2.65) describing the evolution of the density matrix ρ into an equivalent system describing the evolution of its Bloch vector $\boldsymbol{a}$. We perform this transformation for fixed $x \in \Omega$ and interpret $\boldsymbol{a}$ as a simple vector and not as a function.

To see that in fact the evolution of the Bloch coordinates has non-zero contribution only in the component $\boldsymbol{a}$, we calculate the transformed Poisson and Onsager operators $\widehat{\mathcal{J}}(a_0, \boldsymbol{a})$ and $\widehat{\mathcal{K}}(a_0, \boldsymbol{a})$ according to (2.17). Let $(a_0, \boldsymbol{a})$ be the Bloch coordinates of the density matrix ρ, i.e. let $\rho = \widehat{\rho}(\boldsymbol{a})$. Then, we get

$$\widehat{\mathcal{J}}(a_0, \boldsymbol{a}) = \tau^{-1} \circ \mathcal{J}(\widehat{\rho}(\boldsymbol{a})) \circ (\tau^{-1})^*, \qquad \widehat{\mathcal{K}}(a_0, \boldsymbol{a}) = \tau^{-1} \circ \mathcal{K}(\widehat{\rho}(\boldsymbol{a})) \circ (\tau^{-1})^*. \tag{2.73}$$

The choice on the diffeomorphism τ, made in the appendix, yields $(\tau^{-1})^* = \frac{1}{2}\tau$. The linearity of τ, τ^{-1} and $\mathcal{J}(\widehat{\rho}(\boldsymbol{a}))$ yield that for an arbitrary matrix $\tau(b_0, \boldsymbol{b}) = B \in \mathbb{C}^{2\times 2}_{\text{herm}}$, we get from Lemma A.2.1

$$\begin{aligned} \widehat{\mathcal{J}}(a_0, \boldsymbol{a})\big[(b_0, \boldsymbol{b})\big] &= \frac{1}{2}\Big(\tau^{-1} \circ \mathcal{J}(\widehat{\rho}(\boldsymbol{a})) \circ \tau\Big)\big[(b_0, \boldsymbol{b})\big] = \frac{1}{2}\,\tau^{-1}\Big(\frac{i}{\hbar}\big[\widehat{\rho}(\boldsymbol{a}), B\big]\Big) \\ &= \frac{1}{2\hbar}\,\tau^{-1}\Big(\tau\big(0,\, 2(\boldsymbol{a} \times \boldsymbol{b})\big)\Big) = \frac{1}{\hbar}\big(0,\, \boldsymbol{a} \times \boldsymbol{b}\big). \end{aligned} \tag{2.74}$$

Introducing the Bloch coordinates of Q_m by $(\mathrm{q}_0, \mathbf{q}_m) = \tau^{-1}(Q_m)$, we get the following equality by involving Lemma A.2.1 and equation (A.57)

$$
\begin{aligned}
\widehat{\mathcal{K}}(a_0,\boldsymbol{a})\big[(b_0,\boldsymbol{b})\big] &= \Big(\tau^{-1}\circ\mathcal{K}(\widehat{\rho}(\boldsymbol{a}))\circ\tfrac{1}{2}\tau\Big)\big[(b_0,\boldsymbol{b})\big] = \tau^{-1}\Bigg(\sum_{m=1}^{M}\Big[Q_m,\mathcal{C}_{\widehat{\rho}(\boldsymbol{a})}\big[Q_m,\tfrac{1}{2}B\big]\Big]\Bigg)\\
&= -i\,\tau^{-1}\Bigg(\sum_{m=1}^{M}\Big[Q_m,\mathcal{C}_{\widehat{\rho}(\boldsymbol{a})}\Big(\tau\big(0,\,2(\mathbf{q}_m\times\tfrac{1}{2}\boldsymbol{b})\big)\Big)\Big]\Bigg)\\
&= -i\,\tau^{-1}\Bigg(\sum_{m=1}^{M}\Big[Q_m,\tau\Big(\boldsymbol{a}\cdot(\mathbf{q}_m\times\boldsymbol{b}),\,\mathsf{C}_{\boldsymbol{a}}\big(\mathbf{q}_m\times\boldsymbol{b}\big)\Big)\Big]\Bigg)\\
&= -\tau^{-1}\Bigg(\sum_{m=1}^{M}\tau\Big(0,\,2\mathbf{q}_m\times\mathsf{C}_{\boldsymbol{a}}\big(\mathbf{q}_m\times\boldsymbol{b}\big)\Big)\Bigg)\\
&= -2\sum_{m=1}^{M}\Big(0,\,\mathbf{q}_m\times\mathsf{C}_{\boldsymbol{a}}\big(\mathbf{q}_m\times\boldsymbol{b}\big)\Big).
\end{aligned}
\tag{2.75}
$$

In the appendix it is shown that in a pointwise picture, the canonical correlation operator $\mathsf{C}_{\boldsymbol{a}}$ is represented by the matrix $\lambda(|\boldsymbol{a}|)\mathsf{Id}_{3\times3}+\mu(|\boldsymbol{a}|)\,(\boldsymbol{a}\otimes\boldsymbol{a})$. In the present situation, this means

$$
\mathsf{C}_{\boldsymbol{a}}:\begin{cases} L^\infty(\Omega;\mathbb{R}^3_{\boldsymbol{a}}) \longrightarrow (L^1\cap L^\infty)(\Omega;\mathbb{R}^3_{\boldsymbol{a}}),\\ \qquad\qquad \boldsymbol{b}\longmapsto \lambda(|\boldsymbol{a}|)\mathsf{Id}_{3\times3}\,\boldsymbol{b}+\mu(|\boldsymbol{a}|)\,(\boldsymbol{a}\otimes\boldsymbol{a})\boldsymbol{b},\end{cases}
\tag{2.76}
$$

where the functions λ and μ are defined for $|\boldsymbol{a}|<1/2$ by

$$
\lambda(|\boldsymbol{a}|) := \frac{2|\boldsymbol{a}|}{\log(1/2+|\boldsymbol{a}|)-\log(1/2-|\boldsymbol{a}|)},\qquad \mu(|\boldsymbol{a}|) := \frac{1-2\lambda(|\boldsymbol{a}|)}{2|\boldsymbol{a}|^2}.
\tag{2.77}
$$

Furthermore, we have the asymptotics[18]

$$
\lim_{r\to0}\lambda(r)=\frac{1}{2},\qquad \lim_{r\to1/2}\lambda(r)=0 \tag{2.78}
$$

$$
\lim_{r\to0}\mu(r)=\frac{2}{3},\qquad \lim_{r\to1/2}\mu(r)=2. \tag{2.79}
$$

We continuously extend λ and μ to the whole space $\mathbb{R}^3_{\boldsymbol{a}}$ by setting $\lambda(|\boldsymbol{a}|)=0$ for $|\boldsymbol{a}|\geq1/2$. In the following, we denote with $\widehat{\mathcal{J}}(\boldsymbol{a})$ and $\widehat{\mathcal{K}}(\boldsymbol{a})$ the operators defined by

$$
\widehat{\mathcal{J}}(\boldsymbol{a}):\begin{cases}\big(L^1(\Omega;\mathbb{R}^3_{\boldsymbol{a}})\big)^*\longrightarrow L^1(\Omega;\mathbb{R}^3_{\boldsymbol{a}})\\ \qquad\qquad \xi\longmapsto \tfrac{1}{\hbar}\boldsymbol{a}\times\xi\end{cases}
\tag{2.80}
$$

$$
\widehat{\mathcal{K}}(\boldsymbol{a}):\begin{cases}\big(L^1(\Omega;\mathbb{R}^3_{\boldsymbol{a}})\big)^*\longrightarrow L^1(\Omega;\mathbb{R}^3_{\boldsymbol{a}})\\ \qquad\qquad \xi\longmapsto -2\sum_{m=1}^{M}\mathbf{q}_m\times\mathsf{C}_{\boldsymbol{a}}\big(\mathbf{q}_m\times\xi\big).\end{cases}
\tag{2.81}
$$

[18] A plot of the functions λ and μ can be found in Figure 4.1 and Figure 4.2 on page 80.

Next, we transfer the isothermal free energy functional $\mathcal{F}_{\vartheta_*}$ from (2.63) into Bloch coordinates. Due to (2.16) we have $\widehat{\mathcal{F}}_{\vartheta_*}(a_0, \boldsymbol{a}) = \mathcal{F}_{\vartheta_*}(\widehat{\rho}(\boldsymbol{a}))$. Using the rules from Lemma A.2.1 and recalling equation (A.54) we obtain for $\rho = \widehat{\rho}(\boldsymbol{a}) \in \mathcal{R}_2$ the identity

$$\tau^{-1}\big(\mathrm{Tr}(\rho \log \rho)\big) = \frac{1}{4}\log\big(1/4 - |\boldsymbol{a}|^2\big) + \frac{\log\big(1/2 + |\boldsymbol{a}|\big) - \log\big(1/2 - |\boldsymbol{a}|\big)}{2|\boldsymbol{a}|}|\boldsymbol{a}|^2. \quad (2.82)$$

For convenience, we introduce the function $\sigma : [0, 1/2) \longrightarrow [\log(1/4), 0)$ and its derivative

$$\sigma(|\boldsymbol{a}|) := \frac{1}{4}\log\big(1/4 - |\boldsymbol{a}|^2\big) + \frac{|\boldsymbol{a}|^2}{\lambda(|\boldsymbol{a}|)} \quad (2.83)$$

$$\frac{d}{d\boldsymbol{a}}\big(\sigma(|\boldsymbol{a}|)\big) = \frac{\boldsymbol{a}}{\lambda(|\boldsymbol{a}|)}. \quad (2.84)$$

We continuously extend σ to $|\boldsymbol{a}| = 1/2$ by setting $\sigma(1/2) := 0$ and for $|\boldsymbol{a}| > 1/2$ we set $\sigma(|\boldsymbol{a}|) := \infty$. We also extend the derivative $\frac{d}{d\boldsymbol{a}}\big(\sigma(|\boldsymbol{a}|)\big)$ with ∞ for $|\boldsymbol{a}| = 1/2$.

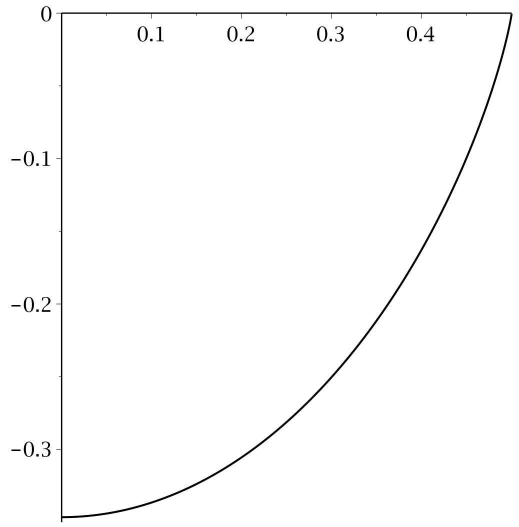

Figure 2.1.: plot of the function σ

Next, we set $(\mathrm{h}_0, \mathbf{h}) = \tau^{-1}(\mathsf{H})$. Then, the isothermal free energy functional $\widehat{\mathcal{F}}_{\vartheta_*}$ in Bloch coordinates[19] is given by

$$\widehat{\mathcal{F}}_{\vartheta_*}(\boldsymbol{a}) = \int_\Omega 2\big(\frac{1}{2}\mathrm{h}_0 + \boldsymbol{a}\cdot\mathbf{h}\big) + 2\,\vartheta_* k_B \sigma(|\boldsymbol{a}|)\,dx \quad (2.85)$$

and consists of the energy functional $\widehat{\mathcal{E}}$ and the entropy functional $\widehat{\mathcal{S}}$ given by

$$\widehat{\mathcal{E}}(\boldsymbol{a}) = \int_\Omega 2\big(\frac{1}{2}\mathrm{h}_0 + \boldsymbol{a}\cdot\mathbf{h}\big)\,dx, \qquad \widehat{\mathcal{S}}(\boldsymbol{a}) = -\int_\Omega 2\,k_B\sigma(|\boldsymbol{a}|)\,dx. \quad (2.86)$$

The derivative of $\widehat{\mathcal{F}}_{\vartheta_*}$ with respect to $\boldsymbol{a}$ is[20]

$$D_{\boldsymbol{a}}\widehat{\mathcal{F}}_{\vartheta_*}(\boldsymbol{a}) = 2\mathbf{h} + 2\,\vartheta_* k_B \frac{\boldsymbol{a}}{\lambda(|\boldsymbol{a}|)}. \quad (2.87)$$

[19] We neglect the dependence on the constant value $a_0 = 1/2$, i.e. we write $\widehat{\mathcal{F}}(\boldsymbol{a})$ instead of $\widehat{\mathcal{F}}(1/2, \boldsymbol{a})$.
[20] We note that the function $D_{\boldsymbol{a}}\widehat{\mathcal{F}}_{\vartheta_*}(\boldsymbol{a}) \in L^\infty(\Omega; \mathbb{R}^3_{\boldsymbol{a}})$ is well defined if $\|\boldsymbol{a}\|_{L^\infty(\Omega)} < 1/2$.

Recalling that $\mathbf{h}$, $\mathbf{q}_m \in L^\infty(\Omega;\mathbb{R}^3_{\boldsymbol{a}})$, for all $m \in \{1,\dots,M\}$, the resulting evolution equation for the Bloch vector $\boldsymbol{a}$ is given by

$$\partial_t \boldsymbol{a} = \frac{1}{\hbar}\boldsymbol{a} \times 2\mathbf{h} + \frac{2}{\vartheta_*}\sum_{m=1}^{M} \mathbf{q}_m \times \mathsf{C}_{\boldsymbol{a}}\Big(\mathbf{q}_m \times \Big(2\mathbf{h} + 2\,\vartheta_* k_B \frac{\boldsymbol{a}}{\lambda(|\boldsymbol{a}|)}\Big)\Big) \qquad \text{in } \Omega. \tag{2.88}$$

Using the "miracle relation" from (2.62) (see also (A.62)) the above equation yields

$$\partial_t \boldsymbol{a} = \frac{1}{\hbar}\boldsymbol{a} \times 2\mathbf{h} + 2\sum_{m=1}^{M} k_B\big(\mathbf{q}_m \times (\mathbf{q}_m \times 2\boldsymbol{a})\big) + \frac{1}{\vartheta_*}\Big(\mathbf{q}_m \times \mathsf{C}_{\boldsymbol{a}}\big(\mathbf{q}_m \times 2\mathbf{h}\big)\Big). \tag{2.89}$$

In the two-level case, the equations (2.88) and (2.89) are equivalent to the evolution equation for the density matrix $\rho \in \mathcal{R}_2$ from (2.69). We end this section with the proof that the Poisson and Onsager operators from (2.80)–(2.81) satisfy (2.1)–(2.2). We stress that the relevant topology of our state space $\mathcal{A}$ is the one induced by the $L^1(\Omega;\mathbb{R}^3_{\boldsymbol{a}})$-norm.

Proposition 2.4.1. *The Poisson operator from* (2.80) *and the Onsager operator from* (2.81) *satisfy the conditions of Definition 2.1.1. This means*

(i) The Poisson operator $\widehat{\mathcal{J}}(\boldsymbol{a})[\xi] := \frac{1}{\hbar}\boldsymbol{a} \times \xi$ *is skew symmetric and satisfies Jacobi's identity, i.e. for all* $\boldsymbol{a} \in \mathcal{A}$ *it holds* $\widehat{\mathcal{J}}(\boldsymbol{a}) = -\widehat{\mathcal{J}}(\boldsymbol{a})^*$ *and for all* $\boldsymbol{a} \in \mathcal{A}$, *for all functionals* $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 : \mathcal{A} \longrightarrow \mathbb{R}$ *the identity*

$$\big\{\{\mathcal{F}_1,\mathcal{F}_2\}_{\widehat{\mathcal{J}}},\mathcal{F}_3\big\}_{\widehat{\mathcal{J}}} + \big\{\{\mathcal{F}_2,\mathcal{F}_3\}_{\widehat{\mathcal{J}}},\mathcal{F}_1\big\}_{\widehat{\mathcal{J}}} + \big\{\{\mathcal{F}_3,\mathcal{F}_1\}_{\widehat{\mathcal{J}}},\mathcal{F}_2\big\}_{\widehat{\mathcal{J}}} \equiv 0 \tag{2.90}$$

is satisfied with $\{\mathcal{F},\mathcal{G}\}_{\widehat{\mathcal{J}}}(\boldsymbol{a}) := \big\langle D_{\boldsymbol{a}}\mathcal{F}(\boldsymbol{a}),\, \widehat{\mathcal{J}}(\boldsymbol{a}) D_{\boldsymbol{a}}\mathcal{G}(\boldsymbol{a})\big\rangle_{L^1(\Omega;\mathbb{R}^3_{\boldsymbol{a}})}$.

(ii) The Onsager operator $\widehat{\mathcal{K}}(\boldsymbol{a})[\xi] := -2\sum_{m=1}^{M} \mathbf{q}_m \times \mathsf{C}_{\boldsymbol{a}}\big(\mathbf{q}_m \times \xi\big)$ *is symmetric and positive semi-definite, i.e. for all* $\boldsymbol{a} \in \mathcal{A}$ *it holds* $\widehat{\mathcal{K}}(\boldsymbol{a}) = \widehat{\mathcal{K}}(\boldsymbol{a})^*$ *and for all* $\boldsymbol{a} \in \mathcal{A}$, *for all* $\xi \in \big(L^1(\Omega;\mathbb{R}^3_{\boldsymbol{a}})\big)^*$ *it holds* $\langle \xi,\, \widehat{\mathcal{K}}(\boldsymbol{a})\,\xi\rangle_{L^1(\Omega;\mathbb{R}^3_{\boldsymbol{a}})} \geq 0$.

Proof. We identify the dual of $L^1(\Omega;\mathbb{R}^3_{\boldsymbol{a}})$ with $L^\infty(\Omega;\mathbb{R}^3_{\boldsymbol{a}})$. For $\boldsymbol{u} \in L^1(\Omega;\mathbb{R}^3_{\boldsymbol{a}})$ and $\boldsymbol{v} \in L^\infty(\Omega;\mathbb{R}^3_{\boldsymbol{a}})$ we identify the dual pairing $\langle \boldsymbol{v},\, \boldsymbol{u}\rangle_{L^1(\Omega;\mathbb{R}^3_{\boldsymbol{a}})}$ with the integral $\int_\Omega \boldsymbol{v}\cdot\boldsymbol{u}\,dx$.
(i) The skew symmetry of $\widehat{\mathcal{J}}(\boldsymbol{a})$ is easy to show, since for all $\boldsymbol{a}$, $\boldsymbol{c} \in L^\infty(\Omega;\mathbb{R}^3_{\boldsymbol{a}})$ and for all $\boldsymbol{b} \in \mathcal{A}$ we have

$$\begin{aligned}\int_\Omega \boldsymbol{a}\cdot\widehat{\mathcal{J}}(\boldsymbol{b})\,\boldsymbol{c}\,dx &= \frac{1}{\hbar}\int_\Omega \boldsymbol{a}\cdot\big(\boldsymbol{b}\times\boldsymbol{c}\big)\,dx = \frac{1}{\hbar}\int_\Omega \boldsymbol{c}\cdot\big(\boldsymbol{a}\times\boldsymbol{b}\big)\,dx\\ &= -\frac{1}{\hbar}\int_\Omega \boldsymbol{c}\cdot\big(\boldsymbol{b}\times\boldsymbol{a}\big)\,dx = -\int_\Omega \boldsymbol{c}\cdot\widehat{\mathcal{J}}(\boldsymbol{b})\,\boldsymbol{a}\,dx.\end{aligned} \tag{2.91}$$

In order to show Jacobi's identity, we denote with $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3 \in L^\infty(\Omega;\mathbb{R}^3_{\boldsymbol{a}})$ the derivatives of some given functionals $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3 : \mathcal{A} \longrightarrow \mathbb{R}$ with respect to $\boldsymbol{a}$ at some given point

$\boldsymbol{a} \in \mathcal{A}$. Then, for j, k, $l \in \{1,2,3\}$ we get from the Lagrange identity for cross products

$$\begin{aligned}\Big\langle \boldsymbol{b}_j,\, D_{\boldsymbol{a}}\widehat{\mathcal{J}}(\boldsymbol{a})\big[\widehat{\mathcal{J}}(\boldsymbol{a})\boldsymbol{b}_l, \boldsymbol{b}_k\big]\Big\rangle_{L^1(\Omega;\mathbb{R}^3_{\boldsymbol{a}})} &= \int_\Omega \boldsymbol{b}_j \cdot \frac{1}{\hbar}\Big(\frac{1}{\hbar}\big(\boldsymbol{a}\times\boldsymbol{b}_l\big)\times\boldsymbol{b}_k\Big)\,dx \\ = \frac{1}{\hbar^2}\int_\Omega \big(\boldsymbol{a}\times\boldsymbol{b}_l\big)\cdot\big(\boldsymbol{b}_k\times\boldsymbol{b}_j\big)\,dx &= \frac{1}{\hbar^2}\int_\Omega \Big(\big(\boldsymbol{a}\cdot\boldsymbol{b}_k\big)\big(\boldsymbol{b}_l\cdot\boldsymbol{b}_j\big) - \big(\boldsymbol{a}\cdot\boldsymbol{b}_j\big)\big(\boldsymbol{b}_l\cdot\boldsymbol{b}_k\big)\Big)\,dx.\end{aligned}$$

Summation over $(j,k,l) \in \{(1,2,3),(2,3,1),(3,1,2)\} =: \mathrm{I}$ yields

$$\begin{aligned}&\sum_{(j,k,l)\in\mathrm{I}} \Big\langle \boldsymbol{b}_j,\, D_{\boldsymbol{a}}\widehat{\mathcal{J}}(\boldsymbol{a})\big[\widehat{\mathcal{J}}(\boldsymbol{a})\boldsymbol{b}_l, \boldsymbol{b}_k\big]\Big\rangle_{L^1(\Omega;\mathbb{R}^3_{\boldsymbol{a}})} = \\ &\frac{1}{\hbar^2}\int_\Omega \Big(\big(\boldsymbol{a}\cdot\boldsymbol{b}_2\big)\big(\boldsymbol{b}_3\cdot\boldsymbol{b}_1\big) - \big(\boldsymbol{a}\cdot\boldsymbol{b}_1\big)\big(\boldsymbol{b}_3\cdot\boldsymbol{b}_2\big) + \big(\boldsymbol{a}\cdot\boldsymbol{b}_3\big)\big(\boldsymbol{b}_1\cdot\boldsymbol{b}_2\big) - \big(\boldsymbol{a}\cdot\boldsymbol{b}_2\big)\big(\boldsymbol{b}_1\cdot\boldsymbol{b}_3\big)\Big)\,dx \\ &+ \frac{1}{\hbar^2}\int_\Omega \Big(\big(\boldsymbol{a}\cdot\boldsymbol{b}_1\big)\big(\boldsymbol{b}_2\cdot\boldsymbol{b}_3\big) - \big(\boldsymbol{a}\cdot\boldsymbol{b}_3\big)\big(\boldsymbol{b}_2\cdot\boldsymbol{b}_1\big)\Big)\,dx = 0. \qquad (2.92)\end{aligned}$$

In view of Lemma 2.1.4 this shows that Jacobi's identity is satisfied.
(ii) It is also an easy task to show the symmetry of $\widehat{\mathcal{K}}(\boldsymbol{a})$. For brevity, we denote the integral $\int_\Omega \boldsymbol{v}\cdot\boldsymbol{u}\,dx$ with $\boldsymbol{v}\cdot\boldsymbol{u}$ in this proof, i.e. we set $\boldsymbol{v}\cdot\boldsymbol{u} := \int_\Omega \boldsymbol{v}\cdot\boldsymbol{u}\,dx$. Since the matrices $\mathsf{Id}_{3\times 3}$ and $\boldsymbol{a}\otimes\boldsymbol{a}$ are symmetric, we have that for all $\boldsymbol{b}\in\mathcal{A}$ and for all $\boldsymbol{a}$, $\boldsymbol{c}\in L^\infty(\Omega;\mathbb{R}^3_{\boldsymbol{a}})$ it holds

$$\begin{aligned}\boldsymbol{a}\cdot\widehat{\mathcal{K}}(\boldsymbol{b})\,\boldsymbol{c} &= -2\sum_{m=1}^M \boldsymbol{a}\cdot\Big(\mathbf{q}_m\times\mathsf{C}_{\boldsymbol{b}}\big(\mathbf{q}_m\times\boldsymbol{c}\big)\Big) = -2\sum_{m=1}^M \mathsf{C}_{\boldsymbol{b}}\big(\mathbf{q}_m\times\boldsymbol{c}\big)\cdot\big(\boldsymbol{a}\times\mathbf{q}_m\big) \\ &= 2\sum_{m=1}^M \big(\mathbf{q}_m\times\boldsymbol{c}\big)\cdot\mathsf{C}_{\boldsymbol{b}}\big(\mathbf{q}_m\times\boldsymbol{a}\big) = 2\sum_{m=1}^M \Big(\mathsf{C}_{\boldsymbol{b}}\big(\mathbf{q}_m\times\boldsymbol{a}\big)\times\mathbf{q}_m\Big)\cdot\boldsymbol{c} \\ &= -2\sum_{m=1}^M \boldsymbol{c}\cdot\Big(\mathbf{q}_m\times\mathsf{C}_{\boldsymbol{b}}\big(\mathbf{q}_m\times\boldsymbol{a}\big)\Big) = \boldsymbol{c}\cdot\widehat{\mathcal{K}}(\boldsymbol{b})\,\boldsymbol{a}. \qquad (2.93)\end{aligned}$$

Also, due to the symmetry of $\mathsf{Id}_{3\times 3}$ and $\boldsymbol{a}\otimes\boldsymbol{a}$, for all $\boldsymbol{a}\in\mathcal{A}$ and for all $\boldsymbol{b}\in L^\infty(\Omega;\mathbb{R}^3_{\boldsymbol{a}})$ we have

$$\boldsymbol{b}\cdot\widehat{\mathcal{K}}(\boldsymbol{a})\,\boldsymbol{b} = -2\sum_{m=1}^M \boldsymbol{b}\cdot\Big(\mathbf{q}_m\times\mathsf{C}_{\boldsymbol{a}}\big(\mathbf{q}_m\times\boldsymbol{b}\big)\Big) = 2\sum_{m=1}^M \big(\mathbf{q}_m\times\boldsymbol{b}\big)\cdot\mathsf{C}_{\boldsymbol{a}}\big(\mathbf{q}_m\times\boldsymbol{b}\big). \qquad (2.94)$$

As shown in Lemma A.2.3, for all $\boldsymbol{a}\in\mathbb{R}^3_{\boldsymbol{a}}$ the matrix $\mathsf{C}_{\boldsymbol{a}}$ is positive semi-definite. This shows the positivity of the operator $\widehat{\mathcal{K}}(\boldsymbol{a})$ for all $\boldsymbol{a}\in\mathcal{A}$ and finishes this proof. □

Due to the above Proposition, it is also clear that the operators from (2.64)–(2.65) satisfy the conditions of Definition 2.1.1 in the case $N = 2$.

2.5. Coupling of Maxwell's Equations to the Dissipative Quantum Model

Next, we give a coupling of Maxwell's equations to the dissipative quantum model in the present context with the coupling mechanisms introduced in Section 1.2. For couplings of the dissipative quantum model from Section 2.4 (or its GENERIC generalization) to other macroscopic systems we refer to [Mie15].

We consider the state space consisting of functions from the space[21]

$$L^2(\Omega;\mathbb{R}^3_{\mathbf{D}}) \times L^2(\Omega;\mathbb{R}^3_{\mathbf{B}}) \times \mathcal{R}_N, \qquad \big(\text{or } L^2(\Omega;\mathbb{R}^3_{\mathbf{D}}) \times L^2(\Omega;\mathbb{R}^3_{\mathbf{B}}) \times \mathcal{A}\big) \tag{2.95}$$

where

$$\Omega \subsetneq \mathbb{R}^3 \quad \text{is a bounded open set}$$

and the sets $\mathcal{R}_N$ (or $\mathcal{A}$) are defined as in Section 2.4 by

$$\mathcal{R}_N := \Big\{\rho \in (L^1 \cap L^\infty)(\Omega;\mathbb{C}^{N\times N}_{\text{herm}}) : \forall_{\text{a.a.}}\, x \in \Omega : \rho(x) \geq 0,\ \text{Tr}\rho(x) = 1\Big\} \tag{2.60}$$

and

$$\mathcal{A} := \Big\{\boldsymbol{a} = (a_1,a_2,a_3) \in (L^1 \cap L^\infty)(\Omega;\mathbb{R}^3_{\boldsymbol{a}}) :\ \|\boldsymbol{a}\|_{L^\infty(\Omega)} \leq 1/2\Big\}. \tag{2.71}$$

We consider a regime, where on the one hand light, described by the electromagnetic field given in the coordinates[22] $\big(\mathbf{D},\mathbf{B}\big) \in L^2(\Omega;\mathbb{R}^3_{\mathbf{D}}) \times L^2(\Omega;\mathbb{R}^3_{\mathbf{B}})$, is influenced by the *active material*, described by the density matrix $\rho \in \mathcal{R}_N$ (or a Bloch vector $\boldsymbol{a} \in \mathcal{A}$). This is modeled via the polarization $\mathbf{P} \in L^2(\Omega;\mathbb{R}^3_{\mathbf{D}})$ given by the constitutive equation[23]

$$\mathbf{P} := \Gamma\rho, \qquad \big(\text{or } \mathbf{P} := \mathsf{G}\boldsymbol{a}\big) \tag{2.96}$$

involving the operator

$$\Gamma \in L^\infty\big(\Omega;\mathcal{L}\big(\mathcal{R}_N, L^2(\Omega;\mathbb{R}^3_{\mathbf{D}})\big)\big), \qquad \big(\text{or } \mathsf{G} \in L^\infty\big(\Omega;\mathcal{L}\big(\mathcal{A}, L^2(\Omega;\mathbb{R}^3_{\mathbf{D}})\big)\big)\big). \tag{2.97}$$

On the other hand, introducing the electric field $\mathbf{E} := \epsilon_0^{-1}\big(\mathbf{D} - \mathbf{P}\big) \in L^2(\Omega;\mathbb{R}^3_{\mathbf{E}})$, our considered regime is such that light influences the active material[24] via the interaction Hamiltonian $\mathsf{H}(\mathbf{E})$ given by the constitutive equation[25]

$$\mathsf{H}(\mathbf{E}) := \Gamma^*\epsilon_0\mathbf{E}, \qquad \big(\text{or } \mathbf{h}(\mathbf{E}) := \mathsf{G}^*\epsilon_0\mathbf{E}\big). \tag{2.98}$$

We stress that G^* is not the polarized dipole moment operator from Chapter 1. The operator Γ^* (or G^*), called the (linear) *dipole moment operator*, is the adjoint of the

[21] As before, the relevant topology for $\mathcal{R}_N$ and $\mathcal{A}$ is the corresponding L^1-topology.
[22] The electric displacement field $\mathbf{D}$ and the magnetic induction $\mathbf{B}$.
[23] Obviously, we have $\mathsf{G} := \Gamma \circ \tau$.
[24] We tacitly assume that the considered material is not magnetizable. Therefore, we set $\mathbf{M} \equiv 0$.
[25] Obviously, we have $\mathsf{G}^* := \tau^* \circ \Gamma^*$.

operator Γ (or G) and given by

$$\Gamma^* \in L^\infty\big(\Omega; \mathcal{L}\big(L^2(\Omega;\mathbb{R}^3_{\mathbf{E}})^*, \mathcal{R}_N^*\big)\big), \qquad \big(\text{or } \mathsf{G}^* \in L^\infty\big(\Omega; \mathcal{L}\big(L^2(\Omega;\mathbb{R}^3_{\mathbf{E}})^*, \mathcal{A}^*\big)\big)\big). \tag{2.99}$$

With the constitutive equations (2.96), (2.98), the total isothermal free energy functional is given as the sum $\mathcal{F}_{\vartheta_*} = \mathcal{E} - \vartheta_* \mathcal{S}$ consisting of the energy functional $\mathcal{E}$ given by

$$\mathcal{E}\big(\mathbf{D}, \mathbf{B}, \rho\big) = \int_\Omega \frac{1}{2\epsilon_0}\big|\mathbf{D} - \Gamma\rho\big|^2 + \frac{1}{2\mu_0}\big|\mathbf{B}\big|^2 + \mathrm{Tr}\big(\rho \mathsf{H}\big)\, dx \tag{2.100}$$

and the entropy functional $\mathcal{S}$ given by

$$\mathcal{S}\big(\mathbf{D}, \mathbf{B}, \rho\big) = -\int_\Omega k_B \mathrm{Tr}\big(\rho \log \rho\big)\, dx. \tag{2.101}$$

Thus, the total isothermal free energy functional is given by

$$\mathcal{F}_{\vartheta_*}\big(\mathbf{D}, \mathbf{B}, \rho\big) = \int_\Omega \frac{1}{2\epsilon_0}\big|\mathbf{D} - \Gamma\rho\big|^2 + \frac{1}{2\mu_0}\big|\mathbf{B}\big|^2 + \mathrm{Tr}\big(\rho \mathsf{H}\big) + \vartheta_* k_B \mathrm{Tr}\big(\rho \log \rho\big)\, dx. \tag{2.102}$$

Its derivative is

$$D\mathcal{F}_{\vartheta_*}\big(\mathbf{D}, \mathbf{B}, \rho\big) = \Big(\epsilon_0^{-1}\big(\mathbf{D} - \Gamma\rho\big),\, \mu_0^{-1}\mathbf{B},\, \big(\epsilon_0^{-1}\Gamma^*\big(\mathbf{D} - \Gamma\rho\big) + \mathsf{H} + \vartheta_* k_B \log \rho\big)\Big). \tag{2.103}$$

The total Poisson operator[26] is given by

$$\mathbb{J}\big(\mathbf{D}, \mathbf{B}, \rho\big) = \begin{pmatrix} 0 & \mathrm{curl} & 0 \\ -\,\mathrm{curl} & 0 & 0 \\ 0 & 0 & \frac{i}{\hbar}\big[\rho, \square\big] \end{pmatrix} \tag{2.104}$$

and the total Onsager operator is given by

$$\mathbb{K}\big(\mathbf{D}, \mathbf{B}, \rho\big) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sum_{m=1}^M \Big[Q^m,\, \mathcal{C}_\rho\big[Q^m,\, \square\big]\Big] \end{pmatrix}. \tag{2.105}$$

Describing the active two-level material with a Bloch vector $\boldsymbol{a} \in \mathcal{A}$, the total isothermal free energy functional $\widehat{\mathcal{F}}_{\vartheta_*}$ consists of the energy functional $\widehat{\mathcal{E}}$ given by

$$\widehat{\mathcal{E}}\big(\mathbf{D}, \mathbf{B}, \boldsymbol{a}\big) = \int_\Omega \frac{1}{2\epsilon_0}\big|\mathbf{D} - \mathsf{G}\boldsymbol{a}\big|^2 + \frac{1}{2\mu_0}\big|\mathbf{B}\big|^2 + 2\big(\frac{1}{2}\mathrm{h}_0 + \boldsymbol{a}\cdot\mathbf{h}\big)\, dx \tag{2.106}$$

and the entropy functional $\widehat{\mathcal{S}}$ given by

$$\widehat{\mathcal{S}}\big(\mathbf{D}, \mathbf{B}, \boldsymbol{a}\big) = -\int_\Omega 2\, k_B \sigma(|\boldsymbol{a}|)\, dx. \tag{2.107}$$

[26]The $\square$-notation was introduced by Öttinger, the symbol $\square$ is a placeholder for the argument.

Thus, the total isothermal free energy functional in Bloch coordinates is given by

$$\widehat{\mathcal{F}}_{\vartheta_*}\big(\mathbf{D},\mathbf{B},\boldsymbol{a}\big) = \int_\Omega \frac{1}{2\epsilon_0}\big|\mathbf{D}-\mathsf{G}\boldsymbol{a}\big|^2 + \frac{1}{2\mu_0}\big|\mathbf{B}\big|^2 + 2\big(\frac{1}{2}\mathrm{h}_0 + \boldsymbol{a}\cdot\mathbf{h}\big) + 2\,\vartheta_* k_B \sigma(|\boldsymbol{a}|)\,dx. \quad (2.108)$$

Its derivative is

$$D\mathcal{F}_{\vartheta_*}\big(\mathbf{D},\mathbf{B},\boldsymbol{a}\big) = \Big(\tfrac{1}{\epsilon_0}\big(\mathbf{D}-\mathsf{G}\boldsymbol{a}\big),\ \tfrac{1}{\mu_0}\mathbf{B},\ \big(\tfrac{1}{\epsilon_0}\mathsf{G}^*\big(\mathbf{D}-\mathsf{G}\boldsymbol{a}\big) + 2\mathbf{h} + 2\,\vartheta_* k_B \frac{\boldsymbol{a}}{\lambda(|\boldsymbol{a}|)}\big)\Big). \quad (2.109)$$

The total Poisson operator is given by

$$\widehat{\mathbb{J}}\big(\mathbf{D},\mathbf{B},\boldsymbol{a}\big) = \begin{pmatrix} 0 & \mathrm{curl} & 0 \\ -\,\mathrm{curl} & 0 & 0 \\ 0 & 0 & \frac{i}{\hbar}\big(\boldsymbol{a}\times\square\big) \end{pmatrix} \quad (2.110)$$

and the total Onsager operator is given by

$$\widehat{\mathbb{K}}\big(\mathbf{D},\mathbf{B},\boldsymbol{a}\big) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2\sum_{m=1}^{M}\mathbf{q}_m \times \mathsf{C}_{\boldsymbol{a}}\big(\mathbf{q}_m\times\square\big) \end{pmatrix}. \quad (2.111)$$

The damped Hamiltonian systems (2.102)–(2.105) and (2.108)–(2.111) have the advantage that the corresponding Poisson and Onsager operators have a somewhat simple structure. For modeling purposes this is desirable. In particular, checking the symmetry properties (2.1) and the structural properties (2.2) of the Poisson and Onsager operators is a somewhat easy task. On the other hand, the disadvantage of the systems in the coordinates $\big(\mathbf{D},\mathbf{B},\rho\big)$ (or $\big(\mathbf{D},\mathbf{B},\boldsymbol{a}\big)$) is that in the resulting evolution equation one gets from (2.3), we would have the variable ρ (or $\boldsymbol{a}$), under the differential operator curl. In the next section we introduce other variables to overcome this analytical difficulty.

We end this section by showing that the operators $\widehat{\mathbb{J}}\big(\mathbf{D},\mathbf{B},\boldsymbol{a}\big)$ and $\widehat{\mathbb{K}}\big(\mathbf{D},\mathbf{B},\boldsymbol{a}\big)$ satisfy the symmetry properties (2.1) and the structural properties (2.2). Since both operators only depend on $\boldsymbol{a}$ we write $\widehat{\mathbb{J}}(\boldsymbol{a})$ and $\widehat{\mathbb{K}}(\boldsymbol{a})$ in the next proposition. Moreover, for brevity we introduce $\mathbf{u} := \big(\mathbf{D},\mathbf{B},\boldsymbol{a}\big)$ and identify $\big(H_0^1(\Omega;\mathbb{R}^3_{\mathbf{D}})\times H_0^1(\Omega;\mathbb{R}^3_{\mathbf{B}})\times L^1(\Omega,\mathbb{R}^3_{\boldsymbol{a}})\big)^*$ with $H_0^1(\Omega;\mathbb{R}^3_{\mathbf{D}})\times H_0^1(\Omega;\mathbb{R}^3_{\mathbf{B}})\times L^\infty(\Omega,\mathbb{R}^3_{\boldsymbol{a}})$. In the following proposition we will not state the image space of any function space, e.g. we write $L^\infty(\Omega)$ instead of $L^\infty(\Omega,\mathbb{R}^3_{\boldsymbol{a}})$.

Proposition 2.5.1. *The Poisson operator $\widehat{\mathbb{J}}(\boldsymbol{a})$ from* (2.110) *and the Onsager $\widehat{\mathbb{K}}(\boldsymbol{a})$ operator from* (2.111) *satisfy the conditions of Definition 2.1.1. This means*

(i) The Poisson operator $\widehat{\mathbb{J}}(\boldsymbol{a}) : H_0^1(\Omega)\times H_0^1(\Omega)\times L^\infty(\Omega) \longrightarrow L^2(\Omega)\times L^2(\Omega)\times L^1(\Omega)$ is skew symmetric and satisfies Jacobi's identity, i.e. for all $\boldsymbol{a}\in\mathcal{A}$ it holds $\widehat{\mathbb{J}}(\boldsymbol{a}) = -\widehat{\mathbb{J}}(\boldsymbol{a})^*$ and for all $\boldsymbol{a}\in\mathcal{A}$, for all functionals $\mathcal{F}_1,\ \mathcal{F}_2,\ \mathcal{F}_3 : H_0^1(\Omega)\times H_0^1(\Omega)\times L^1(\Omega)\longrightarrow\mathbb{R}$ the identity

$$\big\{\{\mathcal{F}_1,\mathcal{F}_2\}_{\widehat{\mathbb{J}}},\mathcal{F}_3\big\}_{\widehat{\mathbb{J}}} + \big\{\{\mathcal{F}_2,\mathcal{F}_3\}_{\widehat{\mathbb{J}}},\mathcal{F}_1\big\}_{\widehat{\mathbb{J}}} + \big\{\{\mathcal{F}_3,\mathcal{F}_1\}_{\widehat{\mathbb{J}}},\mathcal{F}_2\big\}_{\widehat{\mathbb{J}}} \equiv 0 \quad (2.112)$$

is satisfied with $\{\mathcal{F},\mathcal{G}\}_{\widehat{\mathbb{J}}}(\mathbf{u}) := \big\langle D_{\mathbf{u}}\mathcal{F}(\mathbf{u}),\ \widehat{\mathbb{J}}(\boldsymbol{a})D_{\mathbf{u}}\mathcal{G}(\mathbf{u})\big\rangle_{L^2(\Omega)\times L^2(\Omega)\times L^1(\Omega)}$.

(ii) The Onsager operator $\widehat{\mathbb{K}}(\boldsymbol{a})$ is symmetric and positive semi-definite, i.e. for all $\boldsymbol{a} \in \mathcal{A}$ it holds $\widehat{\mathbb{K}}(\boldsymbol{a}) = \widehat{\mathbb{K}}(\boldsymbol{a})^*$ and for all $\boldsymbol{a} \in \mathcal{A}$, for all $\mathbf{u} \in H_0^1(\Omega) \times H_0^1(\Omega) \times L^\infty(\Omega)$ it holds $\langle \mathbf{u}, \widehat{\mathcal{K}}(\boldsymbol{a})\,\mathbf{u}\rangle_{L^2(\Omega)\times L^2(\Omega)\times L^1(\Omega)} \geq 0$.

Proof. Splitting the Poisson operator $\widehat{\mathbb{J}}(\boldsymbol{a})$ into $\widehat{\mathbb{J}}_{\mathrm{alg}}(\boldsymbol{a}) + \widehat{\mathbb{J}}_{\mathrm{ana}}(\boldsymbol{a})$ with

$$\widehat{\mathbb{J}}_{\mathrm{alg}}\big(\boldsymbol{a}\big)[\square] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\hbar}\big(\boldsymbol{a} \times \square\big) \end{pmatrix} \qquad \text{and} \qquad \widehat{\mathbb{J}}_{\mathrm{ana}} = \begin{pmatrix} 0 & \mathrm{curl} & 0 \\ -\,\mathrm{curl} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.113}$$

the statement for the Poisson operator $\widehat{\mathbb{J}}(\boldsymbol{a})$ is immediate due to Proposition 2.3.1 and Proposition 2.4.1, since for all $\boldsymbol{a} \in \mathcal{A}$ we have $\widehat{\mathbb{J}}_{\mathrm{alg}}(\boldsymbol{a}) \circ \widehat{\mathbb{J}}_{\mathrm{ana}}(\boldsymbol{a}) \equiv 0 \equiv \widehat{\mathbb{J}}_{\mathrm{ana}}(\boldsymbol{a}) \circ \widehat{\mathbb{J}}_{\mathrm{alg}}(\boldsymbol{a})$. The statement for the Onsager operator $\widehat{\mathbb{K}}(\boldsymbol{a})$ is immediate due to Proposition 2.4.1. $\square$

2.5.1. Evolution in the Coordinates $\big(\mathbf{E}, \mathbf{H}, \rho\big)$ and $\big(\mathbf{E}, \mathbf{H}, \boldsymbol{a}\big)$

We note that with the *electric field* $\mathbf{E} := \epsilon_0^{-1}\big(\mathbf{D} - \Gamma\rho\big) \in L^2(\Omega; \mathbb{R}^3_{\mathbf{E}})$ and the *magnetic field* $\mathbf{H} := \mu_0^{-1}\mathbf{B} \in L^2(\Omega; \mathbb{R}^3_{\mathbf{H}})$ one can simplify the expression of the isothermal free energy functional $\mathcal{F}_{\vartheta_*}$ from (2.102) (or the corresponding expression from (2.108)). In fact, to avoid the appearance of ρ (or $\boldsymbol{a}$) under the differential operator curl, it is advantageous to use the coordinates $(\mathbf{E}, \mathbf{H}, \rho)$ (or $(\mathbf{E}, \mathbf{H}, \boldsymbol{a})$) instead of $(\mathbf{D}, \mathbf{B}, \rho)$ (or $(\mathbf{D}, \mathbf{B}, \boldsymbol{a})$). This means, we make use of the linear transformation

$$\mathbb{S} : \big(\mathbf{D}, \mathbf{B}, \rho\big) \longmapsto \big(\mathbf{E}, \mathbf{H}, \rho\big), \qquad \mathbb{S} = \begin{pmatrix} \epsilon_0^{-1}\mathsf{Id}_{3\times 3} & 0 & -\epsilon_0^{-1}\Gamma \\ 0 & \mu_0^{-1}\mathsf{Id}_{3\times 3} & 0 \\ 0 & 0 & \mathsf{Id}_{N\times N} \end{pmatrix}, \tag{2.114}$$

or in the Bloch vector case

$$\widehat{\mathbb{S}} : \big(\mathbf{D}, \mathbf{B}, \boldsymbol{a}\big) \longmapsto \big(\mathbf{E}, \mathbf{H}, \boldsymbol{a}\big), \qquad \widehat{\mathbb{S}} = \begin{pmatrix} \epsilon_0^{-1}\mathsf{Id}_{3\times 3} & 0 & -\epsilon_0^{-1}\mathsf{G} \\ 0 & \mu_0^{-1}\mathsf{Id}_{3\times 3} & 0 \\ 0 & 0 & \mathsf{Id}_{3\times 3} \end{pmatrix}. \tag{2.115}$$

We recall the definitions of the Poisson operators from (2.64) and (2.80) given by

$$\mathcal{J}(\rho)[\square] = \frac{i}{\hbar}\big[\rho, \square\big], \qquad \widehat{\mathcal{J}}(\boldsymbol{a})[\square] := \frac{1}{\hbar}\boldsymbol{a} \times \square \tag{2.116}$$

and the definitions of the Onsager operators from (2.65) and (2.81) given by

$$\mathcal{K}(\rho)[\square] = \sum_{m=1}^{M}\Big[Q^m, \mathcal{C}_\rho\big[Q^m, \square\big]\Big], \qquad \widehat{\mathcal{K}}(\boldsymbol{a})[\square] := -2\sum_{m=1}^{M}\mathbf{q}_m \times \mathsf{C}_{\boldsymbol{a}}\big(\mathbf{q}_m \times \square\big). \tag{2.117}$$

The transformed damped Hamiltonian system for the coordinates $\big(\mathbf{E},\mathbf{H},\rho\big)$ is given by the isothermal free energy functional

$$\mathcal{F}_{\vartheta_*}\big(\mathbf{E},\mathbf{H},\rho\big)=\int_\Omega \frac{1}{2}\epsilon_0\big|\mathbf{E}\big|^2+\frac{1}{2}\mu_0\big|\mathbf{H}\big|^2+\mathrm{Tr}(\mathsf{H}\rho)+\vartheta_* k_B\mathrm{Tr}(\rho\log\rho)\,dx, \tag{2.118}$$

the Poisson operator

$$\mathbb{J}\big(\mathbf{E},\mathbf{H},\rho\big)=\epsilon_0^{-1}\begin{pmatrix}\epsilon_0^{-1}\Gamma\mathcal{J}(\rho)[\Gamma^*\Box] & \mu_0^{-1}\,\mathrm{curl} & -\Gamma\mathcal{J}(\rho)[\Box]\\ -\mu_0^{-1}\,\mathrm{curl} & 0 & 0\\ -\mathcal{J}(\rho)[\Gamma^*\Box] & 0 & \epsilon_0\mathcal{J}(\rho)[\Box]\end{pmatrix} \tag{2.119}$$

and the Onsager operator

$$\mathbb{K}\big(\mathbf{E},\mathbf{H},\rho\big)=\epsilon_0^{-1}\begin{pmatrix}\epsilon_0^{-1}\Gamma\mathcal{K}(\rho)[\Gamma^*\Box] & 0 & -\Gamma\mathcal{K}(\rho)[\Box]\\ 0 & 0 & 0\\ -\mathcal{K}(\rho)[\Gamma^*\Box] & 0 & \epsilon_0\mathcal{K}(\rho)[\Box]\end{pmatrix}. \tag{2.120}$$

Inserting the derivative of the free energy functional $\mathcal{F}_{\vartheta_*}(\mathbf{E},\mathbf{H},\rho)$ given by

$$D\mathcal{F}_{\vartheta_*}\big(\mathbf{E},\mathbf{H},\rho\big)=\Big(\epsilon_0\mathbf{E},\ \mu_0\mathbf{H},\ \big(\mathsf{H}+\vartheta_* k_B\log\rho\big)\Big) \tag{2.121}$$

into the general evolution equation for damped Hamiltonian systems (2.3) leads to the following system[27]

$$\partial_t\mathbf{E}=\epsilon_0^{-1}\,\mathrm{curl}\,\mathbf{H}-\epsilon_0^{-1}\Gamma\partial_t\rho \qquad \text{in } \Omega \tag{2.122a}$$

$$\partial_t\mathbf{H}=-\mu_0^{-1}\,\mathrm{curl}\,\mathbf{E} \qquad \text{in } \Omega \tag{2.122b}$$

$$\begin{aligned}\partial_t\rho=&\ \frac{i}{\hbar}\big[\rho,\,\mathsf{H}-\Gamma^*\mathbf{E}\big]\\ &-\frac{1}{\vartheta_*}\sum_{m=1}^{M}\Big[Q^m,\,\mathcal{C}_\rho\big[Q^m,\big(\mathsf{H}-\Gamma^*\mathbf{E}+k_B\vartheta_*\log\rho\big)\big]\Big] \qquad \text{in } \Omega.\end{aligned} \tag{2.122c}$$

Furthermore, in the coordinates $\big(\mathbf{E},\mathbf{H},\boldsymbol{a}\big)$ the isothermal free energy functional $\widehat{\mathcal{F}}_{\vartheta_*}$ consists of the energy functional

$$\widehat{\mathcal{E}}\big(\mathbf{E},\mathbf{H},\boldsymbol{a}\big)=\int_\Omega \frac{1}{2}\epsilon_0\big|\mathbf{E}\big|^2+\frac{1}{2}\mu_0\big|\mathbf{H}\big|^2+2\big(\frac{1}{2}\mathrm{h}_0+\boldsymbol{a}\cdot\mathbf{h}\big)\,dx \tag{2.123}$$

and the entropy functional

$$\widehat{\mathcal{S}}\big(\mathbf{E},\mathbf{H},\boldsymbol{a}\big)=-\int_\Omega 2\,k_B\sigma(|\boldsymbol{a}|)\,dx. \tag{2.124}$$

[27] Here, with a slight abuse of notation, we write $\Gamma^*\mathbf{E}$ instead of $\epsilon_0^{-1}\Gamma^*\epsilon_0\mathbf{E}$.

Thus, in the coordinates $\big(\mathbf{E},\mathbf{H},\boldsymbol{a}\big)$ the isothermal free energy functional $\widehat{\mathcal{F}}_{\vartheta_*}$ is given by

$$\widehat{\mathcal{F}}_{\vartheta_*}\big(\mathbf{E},\mathbf{H},\boldsymbol{a}\big)=\int_\Omega \frac{1}{2}\epsilon_0\big|\mathbf{E}\big|^2+\frac{1}{2}\mu_0\big|\mathbf{H}\big|^2+2\big(\frac{1}{2}\mathrm{h}_0+\boldsymbol{a}\cdot\mathbf{h}\big)+2\,\vartheta_* k_B\sigma(|\boldsymbol{a}|)\,dx. \quad (2.125)$$

In the coordinates $\big(\mathbf{E},\mathbf{H},\boldsymbol{a}\big)$ the Poisson operator is given by

$$\widehat{\mathbb{J}}\big(\mathbf{E},\mathbf{H},\boldsymbol{a}\big)=\epsilon_0^{-1}\begin{pmatrix}\epsilon_0^{-1}\mathsf{G}\widehat{\mathcal{J}}(\boldsymbol{a})[\mathsf{G}^*\Box] & \mu_0^{-1}\operatorname{curl} & -\mathsf{G}\widehat{\mathcal{J}}(\boldsymbol{a})[\Box]\\ -\mu_0^{-1}\operatorname{curl} & 0 & 0\\ -\widehat{\mathcal{J}}(\boldsymbol{a})[\mathsf{G}^*\Box] & 0 & \epsilon_0\widehat{\mathcal{J}}(\boldsymbol{a})[\Box]\end{pmatrix} \quad (2.126)$$

and the Onsager operator is given by

$$\widehat{\mathbb{K}}\big(\mathbf{E},\mathbf{H},\boldsymbol{a}\big)=\epsilon_0^{-1}\begin{pmatrix}\epsilon_0^{-1}\mathsf{G}\widehat{\mathcal{K}}(\boldsymbol{a})[\mathsf{G}^*\Box] & 0 & -\mathsf{G}\widehat{\mathcal{K}}(\boldsymbol{a})[\Box]\\ 0 & 0 & 0\\ -\widehat{\mathcal{K}}(\boldsymbol{a})[\mathsf{G}^*\Box] & 0 & \epsilon_0\widehat{\mathcal{K}}(\boldsymbol{a})[\Box]\end{pmatrix}. \quad (2.127)$$

Inserting the derivative of the free energy functional $\widehat{\mathcal{F}}_{\vartheta_*}(\mathbf{E},\mathbf{H},\rho)$ given by

$$D\widehat{\mathcal{F}}_{\vartheta_*}\big(\mathbf{E},\mathbf{H},\boldsymbol{a}\big)=\Big(\epsilon_0\mathbf{E},\ \mu_0\mathbf{H},\ \big(2\mathbf{h}+2\,\vartheta_* k_B\frac{\boldsymbol{a}}{\lambda(|\boldsymbol{a}|)}\big)\Big) \quad (2.128)$$

into the general evolution equation for damped Hamiltonian systems (2.3) leads to the following system which will be thoroughly studied in Chapter 4

$$\partial_t\mathbf{E}=\epsilon_0^{-1}\operatorname{curl}\mathbf{H}-\epsilon_0^{-1}\mathsf{G}\partial_t\boldsymbol{a} \qquad \text{in } \Omega \quad (2.129a)$$

$$\partial_t\mathbf{H}=-\mu_0^{-1}\operatorname{curl}\mathbf{E} \qquad \text{in } \Omega \quad (2.129b)$$

$$\begin{aligned}\partial_t\boldsymbol{a}=&\frac{1}{\hbar}\boldsymbol{a}\times\big(2\mathbf{h}-\mathsf{G}^*\mathbf{E}\big)\\ &+\frac{2}{\vartheta_*}\sum_{m=1}^{M}\mathbf{q}_m\times\mathsf{C}_{\boldsymbol{a}}\Big(\mathbf{q}_m\times\Big(\big(2\mathbf{h}-\mathsf{G}^*\mathbf{E}\big)+2\,\vartheta_* k_B\frac{\boldsymbol{a}}{\lambda(|\boldsymbol{a}|)}\Big)\Big) \qquad \text{in } \Omega. \quad (2.129c)\end{aligned}$$

Due to Proposition 2.5.1, it is clear, that the operators from (2.126)–(2.127) satisfy the conditions of Definition 2.1.1 and that in the case $N=2$ the operators from (2.119)–(2.120) also satisfy the conditions of Definition 2.1.1.

3. Analysis of a Reduced Maxwell-Bloch Type Model

In Section 2.2 we saw that with minor changes the reduced unidirectional Maxwell-Bloch system (2.23), from Section 1.4, exhibits some structural similarities to damped Hamiltonian systems in the GENERIC framework. Motivated by this system and with the scope of studying a simpler system than (2.129) that has a similar non-linearity and exhibits structural similarities to damped Hamiltonian systems in the GENERIC framework, we arrived at the following system for the amplitude a and the inversion w

$$\partial_t a(x,t) = -\partial_x a(x,t) - \alpha\, w(x,t)\, a(x,t) \tag{2.51a}$$

$$\partial_t w(x,t) = \alpha\, h(w)\, |a(x,t)|^2 - \kappa\, w(x,t). \tag{2.51b}$$

The function $h : \mathbb{R} \longrightarrow [0, 1/2]$ has been defined by[1]

$$h(w) := \frac{w}{f'(w)} = \frac{w}{\log(1+w) - \log(1-w)} \tag{2.48}$$

where f' denotes the extension of the derivative of the free energy density f defined by

$$f'(w) := \begin{cases} -\infty, & w = -1 \\ \log(1+w) - \log(1-w), & w \in (-1,1) \\ +\infty, & w = 1. \end{cases} \tag{2.47}$$

The non-linearity h actually is quite similar to the non-linearity λ appearing in the canonical correlation operator in Bloch coordinates $\mathsf{C}_{\boldsymbol{a}}$ of the dissipative Maxwell-Bloch type model (2.129) from Section 2.5. An interesting difference between the two systems is that (2.129c) is linear in $\mathbf{E}$ whereas (2.51b) is linear in $|a|^2$.

In this chapter we will give a thorough analysis of the above system including an analysis of its long-time behavior. In Section 3.1 we introduce our notion of solutions and give a Lipschitz continuous approximation of the non-Lipschitzean non-linearity h from above. The results are stated in Section 3.2. Section 3.3 to Section 3.5 are devoted to the proofs of the results from Section 3.2. Finally, Section 3.6 contains the analysis of the long-time behavior including the corresponding results.

We highlight that, in contrast to [JoR02] and [LRR07] where similar systems are studied with L^∞-initial data for the amplitude, we can handle initial data for the amplitude from "almost" L^2.

[1] A plot of the function h can be found in Figure 3.1 on page 43.

3.1. Mathematical Formulation of the Problem

In this section we give a mathematical formulation of the Cauchy problem for system (2.51) with given initial data a_0, w_0. As before, we will refer to the function a as *the amplitude* and to the function w as *the inversion*. We understand that our system describes the interaction of a *lasing material* with light, represented by the inversion and the amplitude, respectively. In particular, in the case $\mathbb{T} = \mathbb{S}^1$ our system can be interpreted as the description of a ring laser.

3.1.1. Assumptions on the Data and A-Priori Statements

For a given $T > 0$, we formulate the problem on the space-time domain $[0, T] \times \mathbb{T}$, where $\mathbb{T}$ is a placeholder for either of the sets $\mathbb{R}$ or $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. Furthermore, we assume that in the case $\mathbb{T} = \mathbb{R}$, the lasing material described by the inversion w occupies only a certain finite part of the whole space and is surrounded by vacuum. (The same may hold for the case $\mathbb{T} = \mathbb{S}^1$, but here we also allow the lasing material to occupy the whole torus.) Therefore, we fix the set

$$\Omega_{\text{act}} \subseteq \mathbb{T} \qquad \text{open and bounded} \tag{3.2}$$

which denotes the region occupied by the lasing material. In order that the Poisson operators $\mathbb{J}_{\text{ana}}$ and $\mathbb{J}_{\text{alg}}$ building the operator $\mathbb{J}$ and the Onsager operator $\mathbb{K}$ from (2.49) satisfy the conditions of Definition 2.1.1, we need to impose that $\kappa \geq 0$. Therefore, we fix the two constants

$$\alpha \in \mathbb{R}, \qquad\qquad \kappa \geq 0. \tag{3.3}$$

Furthermore, we assume that the initial data a_0, w_0 is given in such a way that for the free energy functional $\mathcal{F}$ defined by

$$\mathcal{F}(a, w) = \int_{\mathbb{T}} \frac{1}{2}|a|^2 + f(w)\, dx \tag{3.4}$$

with $f(w) = (1 - w)\log(1 - w) + (1 + w)\log(1 + w)$ for $|w| \leq 1$ and $f(w) = \infty$ for $|w| > 1$, we have

$$\mathcal{F}(a_0, w_0) < \infty. \tag{3.5}$$

Moreover, we assume that initially the inversion is only nonzero in the space occupied by the lasing material. For example, these assumptions are satisfied for $(a_0, w_0) \in L^2(\mathbb{T}) \times \mathcal{W}_0$ with

$$\mathcal{W}_0 := \Big\{ w_0 \in \mathrm{L}^\infty(\mathbb{T}) : \ \|w_0\|_{\mathrm{L}^\infty(\mathbb{T})} \leq 1, \ \|w_0\|_{L^\infty(\mathbb{T}\setminus\Omega_{\text{act}})} = 0 \Big\}. \tag{3.6}$$

In order to prevent the vacuum region from lasing in the case $\Omega_{\text{act}} \subsetneq \mathbb{T}$, we need to slightly modify our model from Section 2.2 by introducing the characteristic function $\chi_{\Omega_{\text{act}}}$ of the bounded open set Ω_{act}.

This leads to the system

$$\partial_t a(x,t) = -\partial_x a(x,t) - \alpha\, \chi_{\Omega_{\text{act}}}(x)\, w(x,t)\, a(x,t) \tag{3.9a}$$
$$\partial_t w(x,t) = \alpha\, \chi_{\Omega_{\text{act}}}(x)\, h(w)\, |a(x,t)|^2 - \kappa\, w(x,t). \tag{3.9b}$$

Remark 3.1.1. *In order to keep the structure* (2.3) *we also multiply the term $\alpha\, w\, a$ in* (3.9a) *with the function $\chi_{\Omega_{\text{act}}}$. This means one has to replace the operator $\mathbb{J}$ from* (2.49) *with the operator*

$$\mathbb{J}^{\Omega_{\text{act}}}(a,w) := \begin{pmatrix} -\partial_x & -\chi_{\Omega_{\text{act}}}\gamma(w)\, a \\ \chi_{\Omega_{\text{act}}}\, \gamma(w)\, a & 0 \end{pmatrix}.$$

For the analysis, this is superfluous, though, since for $w_0 \in \mathcal{W}_0$, the evolution equation (3.9b) *ensures $w(t) \equiv 0$ on $\mathbb{T} \setminus \Omega_{\text{act}}$ for all $t \geq 0$. We usually neglect the appearance of $\chi_{\Omega_{\text{act}}}$ in equation* (3.9a).

Taking into account the definition of the function h and the set Ω_{act}, it makes sense to define the set

$$\mathcal{W}_{\text{T}} := \Big\{ w \in W^{1,\infty}((0,T); L^1(\mathbb{T})) : \ \forall\, t \in [0,T] \text{ it holds } w(t) \in \mathcal{W}_0 \Big\}. \tag{3.8}$$

We note that for all $w \in \mathcal{W}_{\text{T}}$ we may infer $w \in C^0([0,T]; L^1(\mathbb{T}))$ due to the continuity of the embedding[2] $W^{1,\infty}((0,T); L^1(\mathbb{T})) \hookrightarrow C^0([0,T]; L^1(\mathbb{T}))$. Our main problem of this chapter is the following.

Problem 3.1.2. *For a given bounded open set $\Omega_{\text{act}} \subseteq \mathbb{T}$, a given final time $T > 0$, given constants $\alpha \in \mathbb{R}$, $\kappa \geq 0$ and given initial data $(a_0, w_0) \in L^2(\mathbb{T}) \times \mathcal{W}_0$, find a couple of functions (a, w) depending on $(x,t) \in [0,T] \times \mathbb{T}$ that satisfies*

$$\partial_t a(x,t) = -\partial_x a(x,t) - \alpha\, \chi_{\Omega_{\text{act}}}(x)\, w(x,t)\, a(x,t) \qquad \text{in } [0,T] \times \mathbb{T} \tag{3.9a}$$
$$\partial_t w(x,t) = \alpha\, \chi_{\Omega_{\text{act}}}(x)\, h(w)\, |a(x,t)|^2 - \kappa\, w(x,t) \qquad \text{in } [0,T] \times \mathbb{T} \tag{3.9b}$$
$$(a(0), w(0)) = (a_0, w_0) \qquad \text{in } \mathbb{T}. \tag{3.9c}$$

Next, we give our notion of weak solutions to (3.9) similar to the one given in [JoR02].

Definition 3.1.3. *A pair of functions $(a,w) \in C^0([0,T]; L^2(\mathbb{T})) \times L^\infty((0,T) \times \mathbb{T})$ is called weak solution to Problem 3.1.2 if the following holds.*

(i) For a.a. $(x,t) \in (0,T) \times \mathbb{T}$ we have

$$w(x,t) = w_0(x) + \int_0^t \alpha\, \chi_{\Omega_{\text{act}}}(x)\, h(w(x,s))\, |a(x,s)|^2 - \kappa\, w(x,s)\, ds. \tag{3.10}$$

[2]See [Zei90, Problem 23.13a, p. 450].

(ii) For all $\varphi \in W^{1,2}(\mathbb{T})$ *and for all* $t \in [0,T]$ *we have*

$$\int_{\mathbb{T}} \varphi(x)\,\big(a(x,t) - a_0(x)\big)\,dx$$
$$= \int_0^t \int_{\mathbb{T}} \big(\partial_x\varphi(x)\,a(x,s) - \alpha\,\varphi(x)\,w(x,s)\,a(x,s)\big)\,dx\,ds. \tag{3.11}$$

The next proposition gives an explanation in which sense a weak solution satisfies the differential equations (3.9) and contains a-priori estimates as well as regularity statements on solutions (a,w) to (3.9) in the above sense.

Proposition 3.1.4. *Let* (a,w) *be a weak solution to Problem 3.1.2 in the sense of Definition 3.1.3. Then it holds*

(i) Equation (3.9a) *is satisfied in the sense of distributions, i.e. for all test functions* $\psi \in C_c^\infty((0,T)\times\mathbb{T})$ *we have*

$$\int_0^T \int_{\mathbb{T}} (\partial_t\psi + \partial_x\psi)\,a(x,s)\,dx\,ds = \alpha \int_0^T \int_{\mathbb{T}} \psi(x,s)\,w(x,s)\,a(x,s)\,dx\,ds. \tag{3.12}$$

Furthermore, for almost all $x \in \mathbb{T}$ *it holds* $a(x,0) = a_0(x)$.

(ii) For all $t \in [0,T]$*, the function* a *satisfies the energy balance*

$$\|a(t)\|^2_{L^2(\mathbb{T})} = \|a_0\|^2_{L^2(\mathbb{T})} - 2\,\alpha \int_0^t \int_{\mathbb{T}} w(x,s)\,|a(x,s)|^2\,dx\,ds. \tag{3.13}$$

(iii) We have $w \in W^{1,\infty}((0,T);L^1(\mathbb{T}))$ *and equation* (3.9b) *is satisfied for almost all* $(x,t) \in (0,T)\times\mathbb{T}$*. Moreover, for a.a.* $x \in \mathbb{T}$ *it holds* $w(x,0) = w_0(x)$.

(iv) The functional $\mathcal{F}$ *from* (3.4) *is absolutely continuous in time* t*. Moreover, the functional* $\mathcal{F}$ *is a Liapunov function in the sense that the following estimates hold*

$$\forall_{\text{a.a.}}\, t \in [0,T]: \qquad \frac{d}{dt}\mathcal{F}(a(t),w(t)) = -\kappa \int_{\mathbb{T}} f'(w(x,t))\,w(x,t)\,dx \le 0 \tag{3.14}$$

$$\forall\, t \in [0,T]: \qquad \mathcal{F}(a(t),w(t)) + \kappa \int_{\mathbb{T}} f'(w(x,t))\,w(x,t)\,dx = \mathcal{F}(a_0,w_0). \tag{3.15}$$

This means the statements of Proposition 2.1.5 hold true for every weak solution (a,w) *to Problem 3.1.2.*

(v) The function w *satisfies*

$$\forall\, t \in [0,T]: \qquad \|w(t)\|_{\mathrm{L}^\infty(\mathbb{T})} \le 1. \tag{3.16}$$

Thus, every weak solution w *to* (3.9b) *satisfies* $w \in \mathcal{W}_{\mathrm{T}}$*. Furthermore, it holds* $w \in C^0([0,T];L^p(\mathbb{T}))$ *for all* $p \in [1,\infty)$.

Proof. (i) We proceed as in [JoR02]. Instead of testing with $\varphi \in W^{1,2}(\mathbb{T})$ in equation (3.11), we take a test function $\partial_t \psi$ with $\psi \in C_c^\infty((0,T)\times\mathbb{T})$. Integrating the resulting equation over $(0,T)$ yields

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{T}} \partial_t \psi(x,t)\, a(t,x)\, dx\, dt = \int_0^T \int_{\mathbb{T}} \partial_t \psi(x,t)\, \big(a(t,x) - a_0(x)\big)\, dx\, dt = \\
&\int_0^T \int_0^t \int_{\mathbb{T}} \Big(\partial_x \partial_t \psi(x,t)\, a(x,s) - \alpha\, \partial_t \psi(x,t)\, w(x,s)\, a(x,s)\Big)\, dx\, ds\, dt = \\
&\int_{\mathbb{T}} \int_0^T \left(\partial_t \partial_x \psi(x,t) \int_0^t a(x,s)\, ds - \alpha\, \partial_t \psi(x,t) \int_0^t w(x,s)\, a(x,s)\, ds\right) dt\, dx = \\
&\int_0^T \int_{\mathbb{T}} \big(- \partial_x \psi(x,t) a(x,t) + \alpha\, \psi(x,t) w(x,t)\, a(x,t)\big)\, dx\, dt.
\end{aligned}
$$

Here, we only used partial integration and the zero boundary condition for ψ as well as $\partial_t a_0(x) = 0$. The second statement is clear due to (3.11).

(ii) We have to postpone this proof to Section 3.3.1.

(iii) Since the integrand in (3.10) belongs to the space $L^\infty((0,T); L^1(\mathbb{T}))$, it holds $w \in W^{1,\infty}((0,T); L^1(\mathbb{T}))$ and $\partial_t w$ is a well defined function in the space $L^\infty((0,T); L^1(\mathbb{T}))$. In particular, this function satisfies $\partial_t w(x,t) = \alpha\, \chi_{\Omega_{\text{act}}}(x)\, h(w(x,t))\, |a(x,t)|^2 - \kappa\, w(x,t)$ for all $t \in [0,T]$ and for almost all $x \in \mathbb{T}$.

(iv) Due to (2.43), (3.13) and (3.10) we have

$$
\begin{aligned}
&\frac{d}{dt} \mathcal{F}(a(t), w(t)) = \frac{d}{dt} \int_{\mathbb{T}} \frac{1}{2} |a(x,t)|^2 + f(w(x,t))\, dx \\
&= -\alpha \int_{\Omega_{\text{act}}} w(x,t)\, |a(x,t)|^2\, dx + \int_{\Omega_{\text{act}}} f'(w(x,t)) \big(\alpha\, h(w(x,t))\, |a(x,t)|^2 - \kappa\, w(x,t)\big)\, dx \\
&= -\kappa \int_{\mathbb{T}} f'(w(x,t))\, w(x,t)\, dx.
\end{aligned}
$$

Since $f'(w)$ and w have the same sign, this implies the estimate $\frac{d}{dt}\mathcal{F}(a(t), w(t)) \leq 0$. Integrating over some time interval $(0,t)$ for an arbitrary $t \in (0,T)$ yields (3.15). The case $\mathbb{K}(a,w) \equiv 0$ in this setting means $\kappa = 0$. In this case $\mathcal{F}(a_0, w_0)$ is obviously a conserved quantity.

(v) To show the a-priori bound, we proceed in three steps. First, we show the a-priori bound for almost all $t \in [0,T]$ in two steps. In the third step, we show that the a-priori bound holds for all $t \in [0,T]$. Afterward, we show the regularity statement. We note that it is clear from equation (3.9b) and our initial data $w_0 \in \mathcal{W}_0$ that we only need to consider the set $\Omega_{\text{act}} \subseteq \mathbb{T}$.
Step 1: We show that for a.a. $(x,t) \in (0,T) \times \Omega_{\text{act}}$ it holds $w(x,t) \geq -1$. To this end, we

assume that there exists a set $N \subset (0,T) \times \Omega_{\text{act}}$ of positive measure with $w(x,t) < -1$ in N. Next, we define $n(x,t) := \min\{w(x,t)+1,\, 0\}$. Then, for a.a. $(x,t) \in (0,T) \times \Omega_{\text{act}}$ it holds $\frac{d}{dt}|n(x,t)|^2 = 2n(x,t) \cdot \partial_t w(x,t)$. Integration over $(0,t) \times \Omega_{\text{act}}$ for some arbitrary $t \in (0,T)$ yields

$$\int_{\Omega_{\text{act}}} \frac{1}{2}|n(x,t)|^2\, dx - \int_{\Omega_{\text{act}}} \int_0^t \big(\alpha\, h(w)\, |a|^2 - \kappa\, w\big)\, n\, ds\, dx = \int_{\Omega_{\text{act}}} \frac{1}{2}|n(x,0)|^2\, dx. \tag{3.17}$$

The right hand side of this equation is obviously equal to zero due to $w_0 \in \mathcal{W}_0$. Furthermore, we have $|n(x,t)|^2 \geq 0$. Thus, the first term on the left hand side is non-negative. For the remaining term we note that on the one hand, we only get a non-zero contribution for those (x,t) satisfying $w(x,t) < -1$. On the other hand, these (x,t) yield $h(w(x,t)) = 0$ as well as $n(x,t) < 0$. Therefore, due to our assumptions on the set N we get

$$-\int_{\Omega_{\text{act}}} \int_0^t \big(\alpha\, h(w)\, |a|^2 - \kappa\, w\big)\, n(x,s)\, ds\, dx = \kappa \int_{\Omega_{\text{act}}} \int_0^t w(x,s)\, n(x,s)\, ds\, dx > 0. \tag{3.18}$$

This yields a contradiction since the right hand side of (3.17) is equal to zero. Therefore, such a set N cannot exist. This proves our lower bound.

Step 2: We show that for a.a. $(x,t) \in (0,T) \times \Omega_{\text{act}}$ it holds $w(x,t) \leq 1$. Now, we assume that there exists a set $P \subset (0,T) \times \Omega_{\text{act}}$ of positive measure with $w(x,t) < 1$ in P and consider the function $p(x,t) := \max\{w(x,t) - 1,\, 0\}$. Again, integrating the equation $\frac{d}{dt}|p(x,t)|^2 = 2p(x,t) \cdot \partial_t w(x,t)$ over $(0,t) \times \Omega_{\text{act}}$ for some arbitrary $t \in (0,T)$ yields

$$\int_{\Omega_{\text{act}}} \frac{1}{2}|p(x,t)|^2\, dx - \int_{\Omega_{\text{act}}} \int_0^t \big(\alpha\, h(w)\, |a|^2 - \kappa\, w\big)\, p\, ds\, dx = \int_{\Omega_{\text{act}}} \frac{1}{2}|p(x,0)|^2\, dx. \tag{3.19}$$

Again, the right hand side of this equation is equal to zero due to $w_0 \in \mathcal{W}_0$. Besides, it holds $|p(x,t)|^2 \geq 0$ and due to our assumptions on the set P, we get

$$-\int_{\Omega_{\text{act}}} \int_0^t \big(\alpha\, h(w)\, |a|^2 - \kappa\, w\big)\, p(x,s)\, ds\, dx = \kappa \int_{\Omega_{\text{act}}} \int_0^t w(x,s)\, p(x,s)\, ds\, dx > 0. \tag{3.20}$$

This also yields a contradiction, since the right hand side of (3.19) is equal to zero. Therefore, such a set P cannot exist. This proves our upper bound as well as the claim for a.a. $t \in [0,T]$.

Step 3: We show that the a-priori bound holds for all $t \in [0,T]$. Due to (iii) and the embedding $W^{1,\infty}((0,T); L^1(\mathbb{T})) \hookrightarrow C^0([0,T]; L^1(\mathbb{T}))$ we have that for all t, $\{t_n\}_{n\in\mathbb{N}} \subset [0,T]$ with $t_n \to t$ it holds

$$\|w(t_n) - w(t)\|_{L^1(\mathbb{T})} \longrightarrow 0 \qquad \text{as } n \to \infty. \tag{3.21}$$

Furthermore, it holds $\|w\|_{L^\infty((0,T);L^\infty(\mathbb{T}))} \leq 1$. Thus, there exists a set $N \subset [0,T]$ of measure zero, such that for all $t \in [0,T] \setminus N$ it holds $\|w(t)\|_{L^\infty(\mathbb{T})} \leq 1$.

For arbitrary $\widetilde{t} \in N$ let some sequence $\{t_n\}_{n\in\mathbb{N}} \subset [0,T] \setminus N$ be given with $t_n \to \widetilde{t}$. Then, it holds $\|w(t_n)\|_{L^\infty(\mathbb{T})} \leq 1$ for all $n \in \mathbb{N}$. This implies the existence of some subsequence $\{t_{n_k}\}_{k\in\mathbb{N}}$ and some $\widetilde{w} \in L^\infty(\mathbb{T})$ with

$$w(t_{n_k}) \longrightarrow \widetilde{w} \qquad \text{weakly-}* \text{ in } L^\infty(\mathbb{T}) \qquad \text{as } k \to \infty. \tag{3.22}$$

Due to the uniqueness of the limit we may infer the equality $\widetilde{w} = w(\widetilde{t})$ from (3.21). Due to the weak-$*$-lower semi-continuity of the norm we may infer $\|w(\widetilde{t})\|_{L^\infty(\mathbb{T})} \leq 1$. Due to the arbitraryness of $\widetilde{t} \in N$, this yields the assertion.

Concluding, we show the regularity statement. For arbitrary $p \in [1,\infty)$ we have

$$\begin{aligned}
\lim_{\vartheta\to 0} \|w(t+\vartheta) - w(t)\|^p_{L^p(\mathbb{T})} &= \lim_{\vartheta\to 0} \int_{\mathbb{T}} |w(t+\vartheta) - w(t)|^{p-1} \cdot |w(t+h) - w(t)| \, dx \\
&\leq \lim_{\vartheta\to 0} \|w(t+\vartheta) - w(t)\|^{p-1}_{L^\infty(\mathbb{T})} \cdot \|w(t+h) - w(t)\|_{L^1(\mathbb{T})} \\
&\leq \lim_{\vartheta\to 0} \left(\|w(t+\vartheta)\|_{L^\infty(\mathbb{T})} + \|w(t)\|_{L^\infty(\mathbb{T})}\right)^{p-1} \cdot \|w(t+\vartheta) - w(t)\|_{L^1(\mathbb{T})} = 0.
\end{aligned} \tag{3.23}$$

This ends the proof of Proposition 3.1.4. □

3.1.2. Regularization

The function h is not Lipschitz-continuous on $[-1,1]$ since its derivative h' tends to $\mp\infty$ for $w \to \pm 1$, as can be anticipated from the plot of the function h in Figure 3.1. This circumstance turned out to make the analysis too difficult to be treated. To

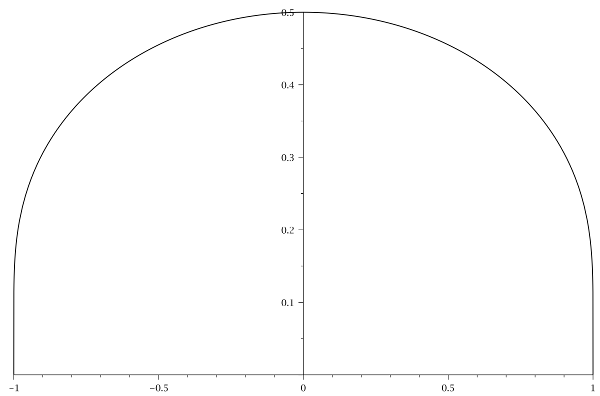

Figure 3.1.: plot of the function h

simplify the analysis for the Cauchy problem (3.9), we create an approximation of system (3.9a)–(3.9b) by replacing h with a linear function if $|w| \geq 1 - \delta$ for some *regularization parameter* $\delta \in [0,1]$ with $\delta \ll 1$. We ask that the resulting function h_δ also satisfies $h_\delta(\pm 1) = 0$. To this end, we calculate the slope m of the linear part of h_δ. It needs to hold

$$-\, m \cdot (1-\delta) + m = \frac{1-\delta}{f'(1-\delta)}. \tag{3.24}$$

This leads to

$$m = m(\delta) = \frac{1-\delta}{\log(\frac{2-\delta}{\delta})\,\delta}. \tag{3.25}$$

Then, we define the regularized function h_δ which is Lipschitz-continuous on $[-1, 1]$ by

$$h_\delta(w) = \begin{cases} m(\delta)\,w + m(\delta), & \text{for } w \in [-1, -(1-\delta)] \\ h(w), & \text{for } w \in [-(1-\delta), (1-\delta)] \\ -m(\delta)\,w + m(\delta), & \text{for } w \in [(1-\delta), 1], \end{cases} \tag{3.26}$$

and continuously extend it with zero for $|w| \geq 1$. The Lipschitz constant of this function is obviously $L_{h_\delta} = m(\delta) = \frac{1-\delta}{\log((2-\delta)/\delta)\,\delta}$.

We aim to get a regularized version of system (3.9) that still exhibits similarities to damped Hamiltonian system in the sense of Definition 2.1.1. To this end, we have to adapt both the functional for the free Energy $\mathcal{F}$ as well as the Operators $\mathbb{J}^{\Omega_{\text{act}}}$ and $\mathbb{K}$ and create regularized versions $\mathcal{F}_\delta$, $\mathbb{J}_\delta^{\Omega_{\text{act}}}$, $\mathbb{K}_\delta$. We still wish to get the relations (2.43) in the regularized versions. Thus, we have to define $\gamma_\delta(w)$, $k_\delta(w)$ and $f_\delta(w)$ in such a way that it holds

$$\gamma_\delta(w) = \alpha\,\frac{w}{f_\delta'(w)}, \qquad k_\delta(w) = \kappa\,\frac{w}{f_\delta'(w)}. \tag{3.27}$$

The functions γ_δ and k_δ can easily be defined in terms of the function h_δ by setting

$$\gamma_\delta(w) := \alpha\,h_\delta(w), \qquad k_\delta(w) := \kappa\,h_\delta(w). \tag{3.28}$$

We ask that the regularized logarithmic potential f_δ also satisfies $f_\delta(0) = 0$ and set[3]

$$f_\delta(w) := \int_0^w \frac{\widetilde{w}}{h_\delta(\widetilde{w})}\,d\widetilde{w}. \tag{3.29}$$

In complience with (2.48), this obviously yields the identity

$$f_\delta'(w) := \frac{w}{h_\delta(w)}. \tag{3.30}$$

Defining the regularized free energy functional $\mathcal{F}_\delta$ by

$$\mathcal{F}_\delta(a, w) = \int_{\mathbb{T}} \frac{1}{2}|a|^2 + f_\delta(w)\,dx \tag{3.31}$$

and the regularized operators $\mathbb{J}_\delta^{\Omega_{\text{act}}}$ and $\mathbb{K}_\delta$ by

$$\mathbb{J}_\delta^{\Omega_{\text{act}}}(a, w) = \begin{pmatrix} -\partial_x & -\chi_{\Omega_{\text{act}}}\gamma_\delta(w)\,a \\ \chi_{\Omega_{\text{act}}}\gamma_\delta(w)\,a & 0 \end{pmatrix}, \qquad \mathbb{K}_\delta(a, w) = \begin{pmatrix} 0 & 0 \\ 0 & k_\delta(w) \end{pmatrix}, \tag{3.32}$$

and plugging this into (2.3), we arrive at the regularized version of system (3.9). In

[3]Note that the function f_δ takes different values at the endpoints $w = \pm 1$ than f.

particular, the regularized analogue to Problem 3.1.2 is stated in the following problem.

Problem 3.1.5. *For a given bounded open set $\Omega_{\text{act}} \subseteq \mathbb{T}$, a given final time $T > 0$, given constants $\alpha \in \mathbb{R}$, $\kappa \geq 0$ and given initial data $(a_0, w_0) \in L^2(\mathbb{T}) \times \mathcal{W}_0$ find a couple of functions (a, w) depending on $(x, t) \in [0, T] \times \mathbb{T}$ that satisfies*

$$\begin{aligned} \partial_t a(x,t) &= -\partial_x a(x,t) - \alpha\, \chi_{\Omega_{\text{act}}}(x)\, w(x,t)\, a(x,t) && \text{in } [0,T] \times \mathbb{T} && (3.33\text{a}) \\ \partial_t w(x,t) &= \alpha\, \chi_{\Omega_{\text{act}}}(x)\, h_\delta(w)\, |a(x,t)|^2 - \kappa\, w(x,t) && \text{in } [0,T] \times \mathbb{T} && (3.33\text{b}) \\ (a(0), w(0)) &= (a_0, w_0) && \text{in } \mathbb{T}. && (3.33\text{c}) \end{aligned}$$

Clearly, Definition 3.1.3 can be adapted to Problem 3.1.5 by replacing h with h_δ. Then, Proposition 3.1.4 holds literally if one replaces h with h_δ and $\mathbb{K}$ with $\mathbb{K}_\delta$.

3.2. Results

In this section, we state our main results, concerning the existence and uniqueness of solutions to the Cauchy problem for both the regularized system (3.33) and the original system (3.9).

Theorem 3.2.1. *For all $T > 0$, an arbitrary bounded open set $\Omega_{\text{act}} \subseteq \mathbb{T}$, arbitrary initial data $(a_0, w_0) \in (L^2 \cap L^\infty)(\mathbb{T}) \times \mathcal{W}_0$ and arbitrary data $\alpha \in \mathbb{R}$, $\kappa \geq 0$, the Cauchy problem*

$$\begin{aligned} \partial_t a(x,t) &= -\partial_x a(x,t) - \alpha\, \chi_{\Omega_{\text{act}}}(x)\, w(x,t)\, a(x,t) && \text{in } [0,T] \times \mathbb{T} && (3.33\text{a}) \\ \partial_t w(x,t) &= \alpha\, \chi_{\Omega_{\text{act}}}(x)\, h_\delta(w)\, |a(x,t)|^2 - \kappa\, w(x,t) && \text{in } [0,T] \times \mathbb{T} && (3.33\text{b}) \\ (a(0), w(0)) &= (a_0, w_0) && \text{in } \mathbb{T} && (3.33\text{c}) \end{aligned}$$

admits a unique weak solution (a, w) in the sense of Definition 3.1.3. Moreover, it holds $(a, w) \in \big(L^\infty((0,T) \times \mathbb{T}) \cap C^0([0,T]; L^p(\mathbb{T}))\big) \times \mathcal{W}_{\text{T}}$ for all $p \in [2, \infty)$ and $\partial_t w \in L^\infty((0,T) \times \mathbb{T})$. In particular, this implies $w \in C^0([0,T]; L^p(\mathbb{T}))$ for all $p \in [1, \infty]$.

The proof of this theorem will be the content of Section 3.3. Using Theorem 3.2.1, we will also be able to prove the following theorem in Section 3.4.

Theorem 3.2.2. *For all $T > 0$ and for every $\sigma > 0$, an arbitrary bounded open set $\Omega_{\text{act}} \subseteq \mathbb{T}$, arbitrary initial data $(a_0, w_0) \in (L^2 \cap L^{2+\sigma})(\mathbb{T}) \times \mathcal{W}_0$ and arbitrary data $\alpha \in \mathbb{R}$, $\kappa \geq 0$, the Cauchy problem*

$$\begin{aligned} \partial_t a(x,t) &= -\partial_x a(x,t) - \alpha\, \chi_{\Omega_{\text{act}}}(x)\, w(x,t)\, a(x,t) && \text{in } [0,T] \times \mathbb{T} && (3.33\text{a}) \\ \partial_t w(x,t) &= \alpha\, \chi_{\Omega_{\text{act}}}(x)\, h_\delta(w)\, |a(x,t)|^2 - \kappa\, w(x,t) && \text{in } [0,T] \times \mathbb{T} && (3.33\text{b}) \\ (a(0), w(0)) &= (a_0, w_0) && \text{in } \mathbb{T} && (3.33\text{c}) \end{aligned}$$

admits a weak solution (a, w) in the sense of Definition 3.1.3 and it holds $(a, w) \in C^0([0,T]; (L^2 \cap L^{2+\sigma})(\mathbb{T})) \times \mathcal{W}_{\text{T}}$.

By means of an a-priori estimate we will also be able to prove a result concerning the Cauchy problem for the original system (3.9), if the considered initial amplitude a_0 is essentially bounded and the modulus of the initial inversion w_0 is bounded away from 1. Namely, in Section 3.5 we will define corresponding sets $\widehat{\mathcal{W}_0}$ and $\widehat{\mathcal{W}_\mathrm{T}}$ with strict inequalities and we will prove the following theorem.

Theorem 3.2.3. *For all $T > 0$, an arbitrary bounded open set $\Omega_{\mathrm{act}} \subseteq \mathbb{T}$, arbitrary initial data $(a_0, w_0) \in (L^2 \cap L^\infty)(\mathbb{T}) \times \widehat{\mathcal{W}_0}$ and arbitrary data $\alpha \in \mathbb{R}$, $\kappa > 0$, the Cauchy problem*

$$\begin{aligned} \partial_t a(x,t) &= -\partial_x a(x,t) - \alpha\, \chi_{\Omega_{\mathrm{act}}}(x)\, w(x,t)\, a(x,t) && \text{in } [0,T]\times\mathbb{T} && (3.9\mathrm{a})\\ \partial_t w(x,t) &= \alpha\, \chi_{\Omega_{\mathrm{act}}}(x)\, h(w)\, |a(x,t)|^2 - \kappa\, w(x,t) && \text{in } [0,T]\times\mathbb{T} && (3.9\mathrm{b})\\ (a(0), w(0)) &= (a_0, w_0) && \text{in } \mathbb{T} && (3.9\mathrm{c}) \end{aligned}$$

has a unique global solution (a, w) in the sense of Definition 3.1.3 and it holds $(a, w) \in \big(L^\infty((0,T)\times\mathbb{T}) \cap C^0([0,T]; L^p(\mathbb{T}))\big) \times \widehat{\mathcal{W}_\mathrm{T}}$ for all $p \in [2,\infty)$.

3.3. A Well-Posedness Proof if $a_0 \in (L^2 \cap L^\infty)$

In this section, we prove Theorem 3.2.1. We recall that $\mathbb{T}$ is a placeholder for either the torus $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ or the set of real numbers $\mathbb{R}$ and that $\Omega_{\mathrm{act}} \subseteq \mathbb{T}$ is the bounded open region occupied by the lasing material. We begin with decoupling the system

$$\partial_t a(x,t) = -\partial_x a(x,t) - \alpha\, \chi_{\Omega_{\mathrm{act}}}(x)\, w(x,t)\, a(x,t) \tag{3.33a}$$
$$\partial_t w(x,t) = \alpha\, \chi_{\Omega_{\mathrm{act}}}(x)\, h_\delta(w)\, |a(x,t)|^2 - \kappa\, w(x,t) \tag{3.33b}$$

and discussing each of the equations separately. Afterward, we will prove Theorem 3.2.1 by means of the Banach fixed point theorem.

3.3.1. The Equation for a

We argue as general as possible. Given an arbitrary final time $T > 0$ and data $\alpha \in \mathbb{R}$, $w \in L^\infty((0,T)\times\mathbb{T})$ we consider the Cauchy problem

$$\begin{aligned} \partial_t a(x,t) &= -\partial_x a(x,t) - \alpha\, \chi_{\Omega_{\mathrm{act}}}(x)\, w(x,t)\, a(x,t) && \text{in } [0,T]\times\mathbb{T} && (3.38\mathrm{a})\\ a(0) &= a_0 && \text{in } \mathbb{T} && (3.38\mathrm{b}) \end{aligned}$$

with $a(0) = a_0 \in (L^2 \cap L^p)(\mathbb{T})$ for arbitrary $p \in [2,\infty]$. We aim to give an existence and uniqueness result as well as an explicit solution formula for the solution to the above Cauchy problem (3.38).

Existence and Uniqueness

We establish an existence and uniqueness result for the Cauchy problem (3.38) with initial data $a_0 \in L^2(\mathbb{T})$ and given $w \in L^\infty((0,T)\times\mathbb{T})$ by showing that for some given

$\widehat{a} \in C^0([0,T]; L^2(\mathbb{T}))$ the solution operator to the Cauchy problem

$$\begin{aligned} \partial_t a(x,t) &= -\partial_x a(x,t) - \alpha\, \chi_{\Omega_{\mathrm{act}}}(x)\, w(x,t)\, \widehat{a}(x,t) && \text{in } [0,T] \times \mathbb{T} && (3.39a) \\ a(0) &= a_0 && \text{in } \mathbb{T}, && (3.39b) \end{aligned}$$

denoted by $\mathcal{T}$, has a unique fixed point in the space $C^0([0,T]; L^2(\mathbb{T}))$. In the following, let us denote the Banach space $L^2(\mathbb{T})$ by X and its dual by X^* as well as the dual pairing by $\langle \cdot, \cdot \rangle_{X^*,X}$. We tacitly identify the dual space X^* with X and the dual pairing $\langle \cdot, \cdot \rangle_{X^*,X}$ with the usual scalar product in $L^2(\mathbb{T})$. However, our notation uses X, X^* and $\langle \cdot, \cdot \rangle_{X^*,X}$. First, we prove the following result concerning the well-posedness of the Cauchy problem (3.39).

Lemma 3.3.1. *For every $\alpha \in \mathbb{R}$, $\widehat{a} \in C^0([0,T]; L^2(\mathbb{T}))$ and $w \in L^\infty((0,T) \times \mathbb{T})$, the Cauchy problem (3.39) has a unique weak solution $a \in C^0([0,T]; X)$ in the sense of Definition A.7.4. In particular, if $\{\mathsf{T}(s)\}_{s \geq 0}$ denotes the semigroup generated by the operator $\frac{d}{dx}$, the unique weak solution to the Cauchy problem (3.39) is given by the variation of constants formula*

$$a(t) = \mathsf{T}(t)\, a_0 - \alpha \int_0^t \mathsf{T}(t-s)\, \chi_{\Omega_{\mathrm{act}}} w(s)\, \widehat{a}(s)\, ds. \tag{3.40}$$

We give a proof based on semigroup theory similar to the proof given in [JoR02, Ch. 4].

Proof. Defining the operator $\mathsf{A} : X \longrightarrow X$ and the function $f_{\widehat{a}} \in L^1((0,T); X)$ by

$$\mathsf{A} := -\frac{d}{dx}, \qquad f_{\widehat{a}}(t) := -\alpha\, \chi_{\Omega_{\mathrm{act}}} w(t)\, \widehat{a}(t), \tag{3.41}$$

we can be interpret the Cauchy problem (3.39) as an initial value problem in the space X by writing

$$\partial_t a(t) = \mathsf{A} a(t) + f_{\widehat{a}}(t), \qquad a(0) = a_0. \tag{3.42}$$

The operator A and its adjoint $\mathsf{A}^* := \frac{d}{dx}$ are well defined in the spaces

$$D(\mathsf{A}) = W^{1,2}(\mathbb{T}), \qquad \text{and} \qquad D(\mathsf{A}^*) = \left(W^{1,2}(\mathbb{T})\right)^* \tag{3.43}$$

and $W^{1,2}(\mathbb{T}) \subset L^2(\mathbb{T})$ is dense. Furthermore, A is a closed operator from $D(\mathsf{A})$ to X (see Lemma A.7.5). Identifying $\left(W^{1,2}(\mathbb{T})\right)^*$ with $W^{1,2}(\mathbb{T})$, i.e. identifying $D(\mathsf{A}^*)$ with $D(\mathsf{A})$, we get that for every $u \in D(\mathsf{A})$ it holds

$$\left\langle u,\, \mathsf{A} u \right\rangle_{X^*,X} = 0 = \left\langle \mathsf{A}^* u,\, u \right\rangle_{X^*,X}. \tag{3.44}$$

Thus, A and A^* are dissipative (see Definition A.7.1). Therefore, by a corollary to the Lumer-Phillips theorem (see Theorem A.7.2), the operator A is the generator of a C_0-semigroup of contractions on X denoted by $\{\mathsf{T}(s)\}_{s \geq 0}$. In particular, this implies $\sup_{t \geq 0} \|\mathsf{T}(t)\|_{\mathcal{L}(X,X)} \leq 1$. From Theorem A.7.3, we may infer that for every $a_0 \in X$, there exists a unique weak solution $a \in C^0([0,T];X)$ to the Cauchy problem (3.39) in the sense of Definition A.7.4 given by the variation of constants formula

$$a(t) = \mathsf{T}(t)\, a_0 + \int_0^t \mathsf{T}(t-s)\, f(s, \widehat{a}(s))\, ds. \tag{3.45}$$

This proves our claim. □

Introducing $L := |\alpha|\, \|w\|_{L^\infty((0,T)\times\mathbb{T})}$, we now show that the solution operator $\mathcal{T} : C^0([0,T];X) \longrightarrow C^0([0,T];X)$ to the Cauchy problem (3.39) defined by

$$\mathcal{T}\widehat{a} := \mathsf{T}(t)a_0 - \alpha \int_0^t \mathsf{T}(t-s)\, \chi_{\Omega_{\mathrm{act}}} w(s)\, \widehat{a}(s)\, ds \tag{3.46}$$

is a contraction on $C^0([0,T];X)$ with respect to the metric[4]

$$d(a_1, a_2) := \max_{t\in[0,T]} \big(e^{-2Lt} \|a_1(t) - a_2(t)\|_X\big). \tag{3.47}$$

To see that $\mathcal{T}$ is a contraction, consider the estimate

$$\begin{aligned}
d\big(\mathcal{T}\widehat{a}_1, \mathcal{T}\widehat{a}_2\big) &= \max_{t\in[0,T]} \Big(e^{-2Lt} \big\|\mathcal{T}\widehat{a}_1 - \mathcal{T}\widehat{a}_2\big\|_X\Big) \\
&\le \max_{t\in[0,T]} \Big(e^{-2Lt} \int_0^t \Big\|\mathsf{T}(t-s)\, \chi_{\Omega_{\mathrm{act}}} \big(\alpha w(s)\widehat{a}_2(s) - \alpha w(s)\widehat{a}_1(s)\big)\Big\|_X\, ds\Big) \\
&\le \max_{t\in[0,T]} \Big(e^{-2Lt}\, |\alpha| \int_0^t \big\|\mathsf{T}(t-s)\big\|_{\mathcal{L}(X,X)} \Big\|w(s)\widehat{a}_2(s) - w(s)\widehat{a}_1(s)\Big\|_X\, ds\Big) \\
&\le \max_{t\in[0,T]} \Big(e^{-2Lt}\, |\alpha|\, \|w\|_{L^\infty((0,T)\times\mathbb{T})} \int_0^t \big\|\widehat{a}_2(s) - \widehat{a}_1(s)\big\|_X\, ds\Big) \\
&\le L \max_{t\in[0,T]} \Big(e^{-2Lt} \int_0^t \big\|\widehat{a}_1(s) - \widehat{a}_2(s)\big\|_X e^{-2Ls} e^{2Ls}\, ds\Big) \\
&\le L\, d(\widehat{a}_1, \widehat{a}_2) \max_{t\in[0,T]} \Big(e^{-2Lt} \int_0^t e^{2Ls}\, ds\Big) \le \frac{1}{2} d(\widehat{a}_1, \widehat{a}_2).
\end{aligned} \tag{3.48}$$

Hence, it holds $L_{\mathcal{T}} = \frac{1}{2}$. The Banach fixed point theorem thus yields the existence of a unique fixed point $a \in C^0([0,T];X)$ of the operator $\mathcal{T}$.

Next, we multiply equation (3.40) for the fixed point a of $\mathcal{T}$ with an arbitrary function $\varphi \in W^{1,2}(\mathbb{T}) = D(\mathsf{A}^*)$ and integrate over $\mathbb{T}$. Then we differentiate the result with respect to time t.

[4]Note that this metric is equivalent to the metric generated by the usual maximum norm $\max_{t\in[0,T]} \|a_1(t) - a_2(t)\|_X$.

This yields the equality

$$\begin{aligned}\frac{d}{dt}\big\langle a(t),\, \varphi\big\rangle_{X,X^*} =& \big\langle \mathsf{T}(t)\, a_0,\, \mathsf{A}^*\varphi\big\rangle_{X,X^*} - \alpha \left\langle \chi_{\Omega_{\mathrm{act}}} w(t)\, a(t),\, \varphi\right\rangle_{X,X^*} \\ & - \alpha \int_0^t \big\langle \mathsf{T}(t-s)\, \chi_{\Omega_{\mathrm{act}}} w(s)\, a(s),\, \mathsf{A}^*\varphi\big\rangle_{X,X^*} ds \\ =& \big\langle a(t),\, \mathsf{A}^*\varphi\big\rangle_{X,X^*} - \alpha \left\langle \chi_{\Omega_{\mathrm{act}}} w(t)\, a(t),\, \varphi\right\rangle_{X,X^*},\end{aligned} \tag{3.49}$$

since it holds $\frac{d\mathsf{T}(t)}{dt} = \mathsf{A}\mathsf{T}(t) = \mathsf{T}(t)\mathsf{A}$ and $\mathsf{T}(0) = \mathsf{Id}$. Integrating this equation over $(0, t)$ for some arbitrary $t \in (0, T)$ yields that $a \in C^0([0, T], X)$ is a weak solution to (3.38) in the sense of Definition 3.1.3. Summarizing, it holds

Lemma 3.3.2. *Given an arbitrary final time $T > 0$, data $\alpha \in \mathbb{R}$, $w \in L^\infty((0, T) \times \mathbb{T})$ and an initial datum $a_0 \in L^2(\mathbb{T})$. Then, there exists a unique weak solution to the Cauchy problem* (3.38) *in the sense of Definition 3.1.3.*

Explicit Solution Formula

In the following, we give an explicit solution formula for the above Cauchy problem (3.38). To derive this formula, we make use of the methods of characteristics. As a first step, we parametrize the initial curve $\mathcal{C}_0$ by

$$\mathcal{C}_0 := \big\{(0, \xi, a_0(\xi)) : \xi \in \mathbb{R}\big\}. \tag{3.50}$$

From equation (3.38) we get the following set of characteristic curves

$$\begin{aligned} t'(s) &= 1, \qquad t(0) = 0, \\ x'(s) &= 1, \qquad x(0) = \xi, \\ z'(s) &= -\alpha\, \chi_{\Omega_{\mathrm{act}}}(x(s))\, w(x(s), s)\, z(s), \qquad z(0) = a_0(\xi). \end{aligned}$$

Solving these equations yields

$$\begin{aligned} t(s) = s, \qquad x(s) = s + \xi, \\ z(s) = a_0(\xi) \cdot \exp\Big(- \alpha \int_0^s \chi_{\Omega_{\mathrm{act}}}(x(\tau))\, w(x(\tau), \tau)\, d\tau\Big). \end{aligned}$$

Insertion of $x(\tau) = \tau + \xi$ for x into the above equation yields

$$z(s) = a_0(\xi) \cdot \exp\Big(- \alpha \int_0^s \chi_{\Omega_{\mathrm{act}}}(\tau + \xi)\, w(\tau + \xi, \tau)\, d\tau\Big).$$

Replacing s and ξ by $t = s$ and $x = \xi + t$, we get the desired explicit solution formula for the amplitude a

$$a(x, t) = a_0(x - t) \cdot \exp\Big(- \alpha \int_0^t \chi_{\Omega_{\mathrm{act}}}(x + \tau - t)\, w(x + \tau - t, \tau)\, d\tau\Big). \tag{3.51}$$

Obviously, by the equation above the function a is well defined in the space $L^\infty((0,T)\times\mathbb{T})$ if $a_0 \in L^\infty(\mathbb{T})$ and $w \in L^1((0,T); L^\infty(\mathbb{T}))$. In the rest of this section we show that the function a defined by (3.51) is a weak solution to (3.38) in the sense of Definition 3.1.3 and prove that this function satisfies the energy balance (3.13) from Proposition 3.1.4.

Lemma 3.3.3. *For given functions $a_0 \in L^2(\mathbb{T})$, $w \in L^\infty((0,T)\times\mathbb{T})$, the function a defined by* (3.51) *is the unique weak solution to* (3.38) *in the sense of Definition 3.1.3.*

Proof. We define the time-shifted functions

$$\widetilde{a}(x,t) := a(x+t,t) = a_0(x)\cdot\exp\Big(-\alpha\int_0^t \chi_{\Omega_{\mathrm{act}}}(x+\tau)\,w(x+\tau,\tau)\,d\tau\Big), \tag{3.52a}$$

$$\widetilde{w}(x,t) := w(x+t,t), \tag{3.52b}$$

$$\widetilde{\chi}_{\Omega_{\mathrm{act}}}(x,t) := \chi_{\Omega_{\mathrm{act}}}(x+t). \tag{3.52c}$$

Then, for all $\psi \in C_c^\infty([0,T]\times\mathbb{T})$ and for all $t \in [0,T]$ it holds

$$\begin{aligned}\int_0^t\int_{\mathbb{T}} a(x,s)\cdot(\partial_t+\partial_x)\,\psi(x,s)\,dx\,ds &= \int_0^t\int_{\mathbb{T}}\widetilde{a}(x,s)\cdot\frac{d}{ds}\psi(x+s,s)\,dx\,ds\\ = -\int_0^t\int_{\mathbb{T}}\frac{d}{ds}\widetilde{a}(x,s)\cdot\psi(x+s,s)\,dx\,ds &+ \int_{\mathbb{T}} a(x,t)\,\psi(x,t) - a_0(x)\,\psi(x,0)\,dx.\end{aligned} \tag{3.53}$$

Insertion of (3.52a) yields that the right hand side of the above equation is equal to

$$\begin{aligned}&= \alpha\int_0^t\int_{\mathbb{T}}\big(\widetilde{\chi}_{\Omega_{\mathrm{act}}}(x,s)\,\widetilde{w}(x,s)\,\widetilde{a}(x,s)\big)\,\psi(x+s,s)\,dx\,ds + \int_{\mathbb{T}}\big(a(t)\,\psi(t) - a_0\,\psi(0)\big)\,dx\\ &= \alpha\int_0^t\int_{\mathbb{T}}\big(\chi_{\Omega_{\mathrm{act}}}(x)\,w(x,s)\,a(x,s)\big)\,\psi(x,s)\,dx\,ds + \int_{\mathbb{T}}\big(a(t)\,\psi(t) - a_0\,\psi(0)\big)\,dx.\end{aligned} \tag{3.54}$$

Choosing a test function ψ that is constant in time, i.e. $\forall\, t \in [0,T] : \psi(x,t) \equiv \varphi(x)$, we get that for all $\varphi \in C_c^\infty(\mathbb{T})$ and for all $t \in [0,T]$ we have

$$\begin{aligned}&\int_{\mathbb{T}}\big(a(x,t) - a_0(x)\big)\,\varphi(x)\,dx\\ &\qquad = \int_0^t\int_{\mathbb{T}}\big(\partial_x\varphi(x)\,a(x,s) - \alpha\,\varphi(x)\,\chi_{\Omega_{\mathrm{act}}}(x)\,w(x,s)\,a(x,s)\big)\,dx\,ds.\end{aligned} \tag{3.55}$$

Due to the density $C_c^\infty(\mathbb{T}) \subset W^{1,2}(\mathbb{T})$, this proves the claim. □

From the explicit solution formula (3.51) for the amplitude a we may actually infer even more. Firstly, since the translation $t \longmapsto a(x-t)$ is a continuous operation[5] in $L^q(\mathbb{T})$ for all $q \in [2,\infty)$, we may infer $a \in C^0([0,T], L^q(\mathbb{T}))$ for every $q \in [2,p] \cap [2,\infty)$.

[5]See [Alt06, Th. 2.14, p. 110].

Secondly, we can directly infer the following estimate for all $q \in [2, p]$

$$\forall\, t \in [0,T]: \qquad \|a(t)\|_{L^q(\mathbb{T})} \leq \|a_0\|_{L^q(\mathbb{T})} \cdot \exp(t\,|\alpha|\,\|w\|_{L^\infty((0,T)\times\mathbb{T})}). \tag{3.56}$$

We summarize our results in

Proposition 3.3.4. *For a fixed time $T > 0$, arbitrary $p \in [2, \infty]$ and $\alpha \in \mathbb{R}$, given an initial datum $a_0 \in (L^2 \cap L^p)(\mathbb{T})$ and a function $w \in L^\infty((0,T) \times \mathbb{T})$, there exists exactly one weak solution a to* (3.38) *in the sense of Definition 3.1.3. In the case $p < \infty$ it holds $a \in C^0([0,T]; L^p(\mathbb{T}))$ and in the case $p = \infty$ it holds $a \in \mathrm{L}^\infty((0,\mathrm{T}) \times \mathbb{T})$. Furthermore, this solution satisfies the following estimate for all $q \in [2, p]$*

$$\forall\, t \in [0,T]: \qquad \|a(t)\|_{L^q(\mathbb{T})} \leq \|a_0\|_{L^q(\mathbb{T})} \cdot \exp(t\,|\alpha|\,\|w\|_{L^\infty((0,T)\times\mathbb{T})}). \tag{3.56}$$

We end this section with giving the proof of Proposition 3.1.4 (ii). Namely, we prove the following lemma that implies Proposition 3.1.4 (ii).

Lemma 3.3.5. *Let $\alpha \in \mathbb{R}$, $w \in L^\infty((0,T) \times \mathbb{T})$ and $a_0 \in L^2(\mathbb{T})$ be given, then, the unique weak solution a to* (3.38) *satisfies*

$$\forall\, t \in [0,T]: \quad \|a(t)\|^2_{L^2(\mathbb{T})} = \|a_0\|^2_{L^2(\mathbb{T})} - 2\alpha \int_0^t \int_{\mathbb{T}} \chi_{\Omega_{\mathrm{act}}}(x)\, w(x,s)\, |a(x,s)|^2\, dx\, ds. \tag{3.13}$$

Proof. From the solution formula (3.51) and the definition of $\widetilde{a}$ in (3.52a) it is clear that for all $t \in (0,T)$ we have

$$\|a(t)\|^2_{L^2(\mathbb{T})} = \|\widetilde{a}(t)\|^2_{L^2(\mathbb{T})} = \left\| a_0(x) \cdot \exp\Big(-\alpha \int_0^t \chi_{\Omega_{\mathrm{act}}}(x+\tau) w(x+\tau)\, d\tau \Big) \right\|_{L^2(\mathbb{T})},$$

since $\mathbb{T}$ is a place holder for $\mathbb{S}^1$ or $\mathbb{R}$. With the notations from (3.52), we get

$$\begin{aligned}
\frac{d}{dt}\|a(t)\|^2_{L^2(\mathbb{T})} &= \frac{d}{dt} \int_{\mathbb{T}} \Big| a_0(x)\, \exp\Big(-\alpha \int_0^t \chi_{\Omega_{\mathrm{act}}}(x+\tau)\, w(x+\tau,\tau)\, d\tau \Big) \Big|^2 dx \\
&= \int_{\mathbb{T}} 2\, a_0(x)\, \exp\Big(-\alpha \int_0^t \chi_{\Omega_{\mathrm{act}}}(x+\tau)\, w(x+\tau,\tau)\, d\tau \Big) \cdot \partial_t \widetilde{a}(x,t)\, dx \\
&= -2\alpha \int_{\mathbb{T}} \widetilde{\chi}_{\Omega_{\mathrm{act}}}(x,t)\, \widetilde{w}(x,t)\, |\widetilde{a}(x,t)|^2\, dx \\
&= -2\alpha \int_{\mathbb{T}} \chi_{\Omega_{\mathrm{act}}}(x)\, w(x,t)\, |a(x,t)|^2\, dx.
\end{aligned} \tag{3.57}$$

Integrating this equality over some arbitrary time interval $(0,t) \subset [0,T]$ yields

$$\|a(t)\|^2_{L^2(\mathbb{T})} = \|a_0\|^2_{L^2(\mathbb{T})} - 2\alpha \int_0^t \int_{\mathbb{T}} \chi_{\Omega_{\mathrm{act}}}(x)\, w(x,s)\, |a(x,s)|^2\, dx\, ds. \tag{3.13}$$

This was the assertion. $\square$

3.3.2. The Equation for w

We consider the integral equation for w given by

$$w(x,t) = w_0(x) + \alpha \int_0^t \chi_{\Omega_{\mathrm{act}}}(x)\, h_\delta(w(x,s))\, |a(x,s)|^2\, ds - \kappa \int_0^t w(x,s)\, ds \tag{3.10}$$

with $w_0 \in \mathcal{W}_0$, given data $\alpha \in \mathbb{R}$, $\kappa \geq 0$ and for a given function $a \in \mathcal{A}_\mathrm{T}$ with

$$\begin{aligned} \mathcal{A}_\mathrm{T} := \Big\{ & a \in \mathrm{L}^\infty((0,\mathrm{T}) \times \mathbb{T}) \;:\; \forall_{\text{a.a.}}\, t \in (0,T) \text{ it holds} \\ & \|a(t)\|_{\mathrm{L}^\infty(\mathbb{T})} \leq e^{|\alpha|\, t} \|a(0)\|_{\mathrm{L}^\infty(\mathbb{T})} \leq e^{|\alpha|\, T} \|a(0)\|_{\mathrm{L}^\infty(\mathbb{T})} \Big\}, \end{aligned} \tag{3.58}$$

$$\mathcal{W}_0 := \Big\{ w_0 \in \mathrm{L}^\infty(\mathbb{T}) \;:\; \|w_0\|_{\mathrm{L}^\infty(\mathbb{T})} \leq 1,\; \|w_0\|_{L^\infty(\mathbb{T}\setminus\Omega_{\mathrm{act}})} = 0 \Big\}. \tag{3.6}$$

For the convenience of the reader, we recall the definition of the set $\mathcal{W}_\mathrm{T}$

$$\mathcal{W}_\mathrm{T} := \Big\{ w \in W^{1,\infty}((0,T); L^1(\mathbb{T})) : \; \forall\, t \in [0,T] \text{ it holds } w(t) \in \mathcal{W}_0 \Big\}. \tag{3.8}$$

Thanks to Proposition 3.1.4 (v), for a given $w_0 \in \mathcal{W}_0$ and $a \in \mathcal{A}_\mathrm{T}$, the operator

$$\mathcal{T}_a : w(x,t) \longmapsto w_0(x) + \alpha \int_0^t \chi_{\Omega_{\mathrm{act}}}(x)\, h_\delta(w(x,s)) |a(x,s)|^2 - \kappa\, w(x,s)\, ds \tag{3.59}$$

maps the following set into itself

$$\Big\{ w \in L^\infty((0,T); L^\infty(\mathbb{T})) : \; \forall_{\text{a.a.}}\, t \in [0,T] \text{ it holds } w(t) \in \mathcal{W}_0 \Big\}. \tag{3.60}$$

Next, we set $c_1(a_0,T) := \|a_0\|_{\mathrm{L}^\infty(\mathbb{T})} \cdot \exp(|\alpha|\, T)$ and define the constant L by

$$L := \max\{ |\alpha| L_{h_\delta} c_1(a_0,T), \kappa \}. \tag{3.61}$$

Since the function h_δ is Lipschitz continuous on $[-1,1]$, the operator $\mathcal{T}_a$ is a contraction on the space (3.60) with respect to the metric

$$d(u,v) := \operatorname*{ess\,sup}_{t\in[0,T]} \|u(t) - v(t)\|_{L^\infty(\mathbb{T})} e^{-2Lt} = \Big\| \|u(t) - v(t)\|_{L^\infty(\mathbb{T})} e^{-2Lt} \Big\|_{L^\infty((0,T))}.$$

Thus, we can infer, as in the Picard-Lindelöf theorem, that the following holds

Proposition 3.3.6. *Let $a \in \mathcal{A}_\mathrm{T}$, $\alpha \in \mathbb{R}$ and $\kappa \geq 0$ be given. Then, for every $w_0 \in \mathcal{W}_0$ and for every $T > 0$, the Cauchy-Problem*

$$\partial_t w(x,t) = \alpha\, \chi_{\Omega_{\mathrm{act}}}(x)\, h_\delta(w)\, |a(x,t)|^2 - \kappa\, w(x,t) \qquad \text{in } [0,T] \times \mathbb{T} \tag{3.62a}$$

$$w(0) = w_0 \qquad \text{in } \mathbb{T} \tag{3.62b}$$

admits a unique solution $w \in W^{1,\infty}((0,T); L^\infty(\mathbb{T})) \cap \mathcal{W}_\mathrm{T}$ in the sense of Definition 3.1.3.

3.3.3. The Coupled Equations

In this section, we prove theorem 3.2.1. To this end, let data $\alpha \in \mathbb{R}$, $\kappa \geq 0$ and initial data $a_0 \in (L^2 \cap L^\infty)(\mathbb{T})$, $w \in \mathcal{W}_0$ be given. Next, we fix some $T > 0$ and introduce the constants

$$C_\alpha(T) := |\alpha| \, \exp(|\alpha| \, T), \qquad c_1(a_0, T) := \|a_0\|_{\mathrm{L}^\infty(\mathbb{T})} \cdot \exp(|\alpha| \, T). \tag{3.63}$$

Then, we define the solution operators

$$\varphi_1 : \mathcal{A}_\mathrm{T} \longmapsto \mathcal{W}_\mathrm{T}, \quad \varphi_1(a) = w, \quad \text{unique solution to (3.62) with IC } w_0, \tag{3.64}$$

$$\varphi_2 : \mathcal{W}_\mathrm{T} \longmapsto \mathcal{A}_\mathrm{T}, \quad \varphi_2(w) = a, \quad \text{unique solution to (3.38) with IC } a_0. \tag{3.65}$$

Then, we show that the map $\Psi := \varphi_2 \circ \varphi_1 : \mathcal{A}_\mathrm{T} \longrightarrow \mathcal{A}_\mathrm{T}$ has a unique fixed point $\widehat{a} \in \mathcal{A}_\mathrm{T}$. To this end, we will make use of the Banach fixed point theorem[6]. From its definition and the discussion of the uncoupled equations, it is clear that Ψ actually maps $\mathcal{A}_\mathrm{T}$ into itself. It remains to show that Ψ is a contraction with respect to the norm $\|\cdot\|_{\mathrm{L}^\infty((0,\mathrm{T})\times\mathbb{T})}$. In fact, we will show that for sufficiently large $k \in \mathbb{N}$, the map Ψ^k is a contraction with respect to this norm.

Estimation of φ_1

Let a_1, $a_2 \in \mathcal{A}_\mathrm{T}$ be given and set $w_1 = \varphi_1(a_1)$, $w_2 = \varphi_1(a_2)$. From the integral equation (3.10) for w and from the Lipschitz continuity of h_δ with Lipschitz constant L_{h_δ}, we get the following estimate for all $t \in [0, T]$

$$\begin{aligned}
&\big\|\varphi_1(a_1)(t) - \varphi_1(a_2)(t)\big\|_{\mathrm{L}^\infty(\mathbb{T})} = \big\|w_1(t) - w_2(t)\big\|_{\mathrm{L}^\infty(\mathbb{T})} \\
&\leq \Big\|\alpha \, \chi_{\Omega_\mathrm{act}} \int_0^t h_\delta(w_1) \, |a_1(s)|^2 - h_\delta(w_2) \, |a_2(s)|^2 \, ds - \kappa \int_0^t w_1(s) - w_2(s) \, ds\Big\|_{\mathrm{L}^\infty(\mathbb{T})} \\
&\leq \Big\|\alpha \, \chi_{\Omega_\mathrm{act}} \int_0^t \big(h_\delta(w_1) - h_\delta(w_2)\big) \, |a_1|^2 + \big(|a_1|^2 - |a_2|^2\big) h_\delta(w_2) \, ds \\
&\qquad - \kappa \int_0^t w_1 - w_2 \, ds\Big\|_{\mathrm{L}^\infty(\mathbb{T})} \\
&\leq \Big\|\alpha \int_0^t \Big(L_{h_\delta}\big(w_1 - w_2\big) \, |a_1|^2 + \big(|a_1|^2 - |a_2|^2\big)\frac{1}{2}\Big) \, ds - \kappa \int_0^t w_1 - w_2 \, ds\Big\|_{\mathrm{L}^\infty(\mathbb{T})}.
\end{aligned} \tag{3.66}$$

Since a_1, $a_2 \in \mathcal{A}_\mathrm{T}$ we have $|a_1|$, $|a_2| \leq c_1(a_0, T)$ a.e. in $Q_T := [0, T] \times \mathbb{T}$. Furthermore, for all $|u| \leq c_1(a_0, T)$ the function $u \mapsto |u|^2$ is Lipschitz-continuous with Lipschitz constant $L_2 \leq 2 \, c_1(a_0, T) = 2 \, \|a_0\|_{\mathrm{L}^\infty(\mathbb{T})} \cdot \exp(|\alpha| \, T)$. With this and the triangle inequality, we can

[6] Namely, the Banach fixed point theorem in the version of [ZeW86, Th. 17A, p. 724].

infer the following estimate from (3.66)

$$\begin{aligned}
\big\|\varphi_1(a_1)(t) - \varphi_1(a_2)(t)\big\|_{\mathrm{L}^\infty(\mathbb{T})} &= \big\|w_1(t) - w_2(t)\big\|_{\mathrm{L}^\infty(\mathbb{T})} \\
&\leq |\alpha| \int_0^t L_{h_\delta}\, c_1^2(a_0,T)\big\|w_1 - w_2\|_{\mathrm{L}^\infty(\mathbb{T})} + \frac{1}{2}\big\||a_1|^2 - |a_2|^2\big\|_{\mathrm{L}^\infty(\mathbb{T})}\, ds \\
&\quad + \kappa \int_0^t \big\|w_1 - w_2\big\|_{\mathrm{L}^\infty(\mathbb{T})}\, ds \\
&\leq c_2 \int_0^t \big\|(w_1 - w_2)(s)\|_{\mathrm{L}^\infty(\mathbb{T})}\, ds + \frac{L_2}{2}\int_0^t \big\|(a_1 - a_2)(s)\big\|_{\mathrm{L}^\infty(\mathbb{T})}\, ds \qquad (3.67)
\end{aligned}$$

with $c_2 := 2\max\{|\alpha| L_{h_\delta}\, c_1^2(a_0,T), \kappa\}$. Gronwall's lemma yields the following estimate for all $t \in [0,T]$ from (3.67)

$$\big\|w_1(t) - w_2(t)\big\|_{\mathrm{L}^\infty(\mathbb{T})} \leq \exp(c_2\, t) \cdot c_1(a_0,T) \int_0^t \big\|(a_1 - a_2)(s)\big\|_{\mathrm{L}^\infty(\mathbb{T})}\, ds. \qquad (3.68)$$

In particular this implies

$$\big\|w_1 - w_2\big\|_{\mathrm{L}^\infty((0,\mathrm{T})\times\mathbb{T})} \leq \exp(c_2\, T) \cdot c_1(a_0,T) \int_0^T \big\|(a_1 - a_2)(s)\big\|_{\mathrm{L}^\infty(\mathbb{T})}\, ds. \qquad (3.69)$$

Estimation of φ_2

Let $w_1,\ w_2 \in \mathcal{W}_\mathrm{T}$ be the solutions corresponding to a_1 and a_2 respectively, i.e. $w_1 = \varphi_1(a_1),\ w_2 = \varphi_1(a_2)$. Furthermore, we set $\widetilde{a}_1 = \varphi_2(w_1),\ \widetilde{a}_2 = \varphi_2(w_2)$ and recall $c_1(a_0,T) := \|a_0\|_{\mathrm{L}^\infty(\mathbb{T})} \cdot \exp(|\alpha|\, T)$. From the section on the equation for w we know that for a.a. $(x,t) \in (0,T) \times \mathbb{T}$ it holds $|w_i(x,t)| \leq 1$, for $i \in \{1,2\}$. Thus, writing $(x + \tau - t, \tau) =: (\xi_\tau^t, \tau)$ and taking advantage of the estimate from Lemma A.1.13, we may infer the following estimate for all $t \in [0,T]$ from the solution formula (3.51) for a (note that due to this solution formula the following calculations are rigorous)

$$\begin{aligned}
&\big\|\varphi_2(w_1)(t) - \varphi_2(w_2)(t)\big\|_{\mathrm{L}^\infty(\mathbb{T})} = \big\|\widetilde{a}_1(t) - \widetilde{a}_2(t)\big\|_{\mathrm{L}^\infty(\mathbb{T})} \\
&\leq \Big\|a_0(x-t) \cdot \Big|\exp\big(-\int_0^t \alpha\, w_1(\xi_\tau^t,\tau)\, d\tau\big) - \exp\big(-\int_0^t \alpha\, w_2(\xi_\tau^t,\tau)\, d\tau\big)\Big|\Big\|_{\mathrm{L}^\infty(\mathbb{T})} \\
&\leq \|a_0\|_{\mathrm{L}^\infty(\mathbb{T})}\, C_\alpha(T) \cdot \int_0^t \Big\|w_1(\tau + x - t, \tau) - w_2(\tau + x - t, \tau)\Big\|_{\mathrm{L}^\infty(\mathbb{T})}\, d\tau \\
&\leq |\alpha|\, c_1(a_0,T) \int_0^t \big\|w_1(\tau) - w_2(\tau)\big\|_{\mathrm{L}^\infty(\mathbb{T})}\, d\tau. \qquad (3.70)
\end{aligned}$$

Ψ^k is a contraction

From the above, it makes sense to set $\widetilde{a}_1 := \Psi(a_1)$ and $\widetilde{a}_2 := \Psi(a_2)$ and to consider the $\mathrm{L}^\infty(\mathbb{T})$-norm of the difference $\Psi(a_1) - \Psi(a_2)$ at some arbitrary point $t \in (0,T)$.

Next, we insert (3.68) into (3.70). This yields

$$\begin{aligned}
&\big\|\Psi(a_1)(t) - \Psi(a_2)(t)\big\|_{\mathrm{L}^\infty(\mathbb{T})} = \big\|\widetilde{a}_1(t) - \widetilde{a}_2(t)\big\|_{\mathrm{L}^\infty(\mathbb{T})} \\
&\le |\alpha|\, c_1(a_0,T) \int_0^t \exp(c_2\,\tau) \cdot c_1(a_0,T) \int_0^\tau \big\|(a_1 - a_2)(s)\big\|_{\mathrm{L}^\infty(\mathbb{T})}\, ds\, d\tau \\
&\le c_3(T) \int_0^t \int_0^\tau \big\|(a_1 - a_2)(s)\big\|_{\mathrm{L}^\infty(\mathbb{T})}\, ds\, d\tau
\end{aligned} \tag{3.71}$$

with $c_3(T) := |\alpha|\, c_1^2(a_0,T)\, \exp(c_2\, T)$. Clearly, we also have the following estimate for all $t \in [0,T]$

$$\big\|\Psi(a_1)(t) - \Psi(a_2)(t)\big\|_{\mathrm{L}^\infty(\mathbb{T})} \le c_3(T) \int_0^t \tau\, \big\|a_1 - a_2\big\|_{L^\infty([0,\tau]\times\mathbb{T})}\, d\tau. \tag{3.72}$$

In particular, for all $t \in [0,T]$ it holds

$$\big\|\Psi(a_1)(t) - \Psi(a_2)(t)\big\|_{\mathrm{L}^\infty(\mathbb{T})} \le c_3(T) \int_0^t \tau\, \big\|a_1 - a_2\big\|_{L^\infty((0,t)\times\mathbb{T})}\, d\tau. \tag{3.73}$$

Thus, for $k = 1$ and for all $t \in [0,T]$ we have shown the validity of the estimate

$$\big\|\Psi^k(a_1)(t) - \Psi^k(a_2)(t)\big\|_{\mathrm{L}^\infty(\mathbb{T})} \le \frac{c_3^k(T)}{(2k-1)!} \int_0^t \tau^{2k-1}\, \big\|a_1 - a_2\big\|_{L^\infty((0,t)\times\mathbb{T})}\, d\tau. \tag{3.74}$$

By induction, this estimate holds for all $k \in \mathbb{N}$, since we have

$$\begin{aligned}
&\big\|\Psi^{k+1}(a_1)(t) - \Psi^{k+1}(a_2)(t)\big\|_{\mathrm{L}^\infty(\mathbb{T})} \\
&\le c_3(T) \int_0^t \int_0^\tau \big\|(\Psi^k(a_1) - \Psi^k(a_2))(s)\big\|_{\mathrm{L}^\infty(\mathbb{T})}\, ds\, d\tau \qquad \text{due to (3.71)} \\
&\le c_3(T) \int_0^t \int_0^\tau \left(\frac{c_3^k(T)}{(2k-1)!} \int_0^s \sigma^{2k-1}\, \big\|a_1 - a_2\big\|_{L^\infty((0,t)\times\mathbb{T})}\, d\sigma \right) ds\, d\tau \\
&= \frac{c_3^{k+1}(T)}{(2k-1)!} \int_0^t \int_0^\tau \frac{s^{2k}}{2k} \big\|a_1 - a_2\big\|_{L^\infty((0,t)\times\mathbb{T})}\, ds\, d\tau \\
&= \frac{c_3^{k+1}(T)}{(2k)!} \int_0^t \frac{\tau^{2k+1}}{2k+1} \big\|a_1 - a_2\big\|_{L^\infty((0,t)\times\mathbb{T})}\, d\tau \\
&= \frac{c_3^{k+1}(T)}{(2(k+1)-1)!} \int_0^t \tau^{2(k+1)-1}\, \big\|a_1 - a_2\big\|_{L^\infty((0,t)\times\mathbb{T})}\, d\tau.
\end{aligned} \tag{3.75}$$

Therefore, for all $k \in \mathbb{N}$ we get the following estimate from (3.74) for $t = T$

$$\big\|\Psi^k(a_1) - \Psi^k(a_2)\big\|_{\mathrm{L}^\infty((0,\mathrm{T})\times\mathbb{T})} \le \frac{T^{2k} c_3^k(T)}{(2k)!}\, \big\|a_1 - a_2\big\|_{\mathrm{L}^\infty((0,\mathrm{T})\times\mathbb{T})}. \tag{3.76}$$

Since the faculty grows faster than the exponential function to any power, for sufficiently large $k \in \mathbb{N}$ we have $\frac{T^{2k} c_3^k(T)}{(2k)!} < 1$. Thus, for sufficiently large $k \in \mathbb{N}$, the map Ψ^k is a contraction and has a unique fixed point $\widehat{a} \in \mathcal{A}_{\mathrm{T}}$. The unique global solution (a, w) of the system (3.33) is now given by $a = \widehat{a}$ and $w = \varphi_1(\widehat{a})$. Due to Proposition 3.3.4, Proposition 3.3.6 and the embedding[7] $W^{1,\infty}((0,T); L^\infty(\mathbb{T})) \hookrightarrow C^0([0,T]; L^\infty(\mathbb{T}))$ this proves Theorem 3.2.1.

3.4. An Existence Proof if $a_0 \in (L^2 \cap L^{2+\sigma})$

In this section, we discuss the existence issue of a solution to the Cauchy problem

$$\begin{aligned}
\partial_t a(x,t) &= -\partial_x a(x,t) - \alpha\, \chi_{\Omega_{\mathrm{act}}}(x)\, w(x,t)\, a(x,t) && \text{in } [0,T]\times\mathbb{T} && \text{(3.33a)}\\
\partial_t w(x,t) &= \alpha\, \chi_{\Omega_{\mathrm{act}}}(x)\, h_\delta(w)\, |a(x,t)|^2 - \kappa\, w(x,t) && \text{in } [0,T]\times\mathbb{T} && \text{(3.33b)}\\
(a(0), w(0)) &= (a_0, w_0) && \text{in } \mathbb{T} && \text{(3.33c)}
\end{aligned}$$

with initial data (a_0, w_0) from the more general space $(L^2 \cap L^{2+\sigma})(\mathbb{T}) \times \mathcal{W}_0$ with

$$\mathcal{W}_0 := \Big\{ w_0 \in \mathrm{L}^\infty(\mathbb{T}) :\ \|w_0\|_{\mathrm{L}^\infty(\mathbb{T})} \le 1,\ \|w_0\|_{L^\infty(\mathbb{T}\setminus\Omega_{\mathrm{act}})} = 0 \Big\} \tag{3.6}$$

and prove Theorem 3.2.2. We recall that $\sigma > 0$ is arbitrary and note that again $\mathbb{T}$ is a placeholder for either of the sets $\mathbb{R}$ or $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ and that again $\Omega_{\mathrm{act}} \subseteq \mathbb{T}$ is the bounded open set occupied by the lasing material. Again, we set $Q_T := [0,T] \times \mathbb{T}$ and $\Omega_T := [0,T] \times \Omega_{\mathrm{act}}$.

3.4.1. Approximation

We begin with fixing some arbitrary final time $T > 0$ for the rest of this section and approximate the initial data $(a_0, w_0) \in (\mathrm{L}^2 \cap \mathrm{L}^{2+\sigma})(\mathbb{T}) \times \mathcal{W}_0$ with functions from $(\mathrm{L}^2 \cap \mathrm{L}^\infty)(\mathbb{T}) \times \mathcal{W}_0$. To this end, we fix some representant of a_0 and define for all $\lambda \ge 1$

$$a_0^\lambda(x) := \max\big\{ -\lambda, \min\{a_0(x), \lambda\}\big\}. \tag{3.78}$$

Then, we obviously have $a_0^\lambda \in (\mathrm{L}^2 \cap \mathrm{L}^\infty)(\mathbb{T})$ with $\|a_0^\lambda\|_{\mathrm{L}^\infty(\mathbb{T})} \le \lambda$ as well as

$$a_0^\lambda \longrightarrow a_0 \qquad \text{strongly in } (L^2 \cap L^{2+\sigma})(\mathbb{T}). \tag{3.79}$$

We recall the definition of the set $\mathcal{W}_{\mathrm{T}}$ given by

$$\mathcal{W}_{\mathrm{T}} := \Big\{ w \in W^{1,\infty}((0,T); L^1(\mathbb{T})) :\ \forall\, t \in [0,T] \text{ it holds } w(t) \in \mathcal{W}_0 \Big\}. \tag{3.8}$$

[7] See [Zei90, Problem 23.13a, p. 450].

From Theorem 3.2.1 we get that for all $\lambda \geq 1$ there exists a unique solution $(a_\lambda, w_\lambda) \in \big(L^\infty((0,T)\times\mathbb{T}) \cap C^0([0,T];L^p(\mathbb{T}))\big) \times \mathcal{W}_{\mathrm{T}}$ for all $p \in [2,\infty)$ to the Cauchy problem

$$\partial_t a(x,t) = -\partial_x a(x,t) - \alpha\,\chi_{\Omega_{\mathrm{act}}}(x)\,w(x,t)\,a(x,t) \qquad \text{in } [0,T]\times\mathbb{T} \tag{3.80a}$$
$$\partial_t w(x,t) = \alpha\,\chi_{\Omega_{\mathrm{act}}}(x)\,h_\delta(w)\,|a(x,t)|^2 - \kappa\,w(x,t) \qquad \text{in } [0,T]\times\mathbb{T} \tag{3.80b}$$
$$(a(0),w(0)) = (a_0^\lambda, w_0) \qquad \text{in } \mathbb{T} \tag{3.80c}$$

with initial data $(a_0^\lambda, w_0) \in (\mathrm{L}^2 \cap \mathrm{L}^\infty)(\mathbb{T}) \times \mathcal{W}_0$. Moreover, for all $\lambda \geq 1$ we have $w_\lambda \in C^0([0,T];L^p(\mathbb{T}))$ for all $p \in [1,\infty)$. Furthermore, we a-priorily have the following bounds.

Lemma 3.4.1. *Let $\{(a_\lambda, w_\lambda)\}_{\lambda\geq 1}$ be the family of unique solutions to the Cauchy problem* (3.80)*. Then, there exist constants C^a_{uni} and C^w_{uni} that are independent of $\lambda \geq 1$, such that for all $t \in [0,T]$ and for all $\lambda \geq 1$ the following bounds hold true*

$$\big\|w_\lambda(t)\big\|_{L^\infty(\mathbb{T})} \leq 1, \qquad \big\|w_\lambda(t)\big\|_{L^2(\mathbb{T})} \leq C^w_{\mathrm{uni}} \tag{3.81a}$$
$$\big\|a_\lambda(t)\big\|_{L^2(\mathbb{T})} \leq C^a_{\mathrm{uni}}\big\|a_0\big\|_{L^2(\mathbb{T})}, \qquad \big\|a_\lambda(t)\big\|_{L^{2+\sigma}(\mathbb{T})} \leq C^a_{\mathrm{uni}}\big\|a_0\big\|_{L^{2+\sigma}(\mathbb{T})}. \tag{3.81b}$$

Proof. The bounds in (3.81a) are clear due to Proposition 3.1.4 and the boundedness of Ω_{act}. To see the bounds in (3.81b), consider the solution formula for a_λ given by (3.51) and take the $L^p(\mathbb{T})$-norm for $p \in \{2,(2+\sigma)\}$. For all $\lambda \geq 1$ this yields

$$\forall\, t \in [0,T]: \qquad \big\|a_\lambda(t)\big\|_{L^p(\mathbb{T})} \leq \big\|a_0^\lambda\big\|_{L^p(\mathbb{T})} \exp\big(|\alpha|\,t\,\big\|w_\lambda\big\|_{\mathrm{L}^\infty((0,\mathrm{T})\times\mathbb{T})}\big) \tag{3.82}$$

for $p \in \{2,(2+\sigma)\}$. Due to the bounds (3.81a) and due to $\sup_{\lambda\geq1}\|a_0^\lambda\|_{L^p(\mathbb{T})} \leq \|a_0\|_{L^p(\mathbb{T})}$ for $p \in \{2,(2+\sigma)\}$, we get the claim with $C^a_{\mathrm{uni}} = \exp\big(|\alpha|\,T\big)$. ☐

Formally multiplying the equation for a_λ with a_λ suggests that $|a_\lambda|^2$ is a solution to

$$\partial_t|a_\lambda|^2 + \partial_x|a_\lambda|^2 = -2\,\alpha\,\chi_{\Omega_{\mathrm{act}}}\,w_\lambda\,|a_\lambda|^2. \tag{3.83}$$

The next lemma justifies this suggestion.

Lemma 3.4.2. *For an arbitrary $\lambda \geq 1$, let $(a_\lambda, w_\lambda) \in C^0([0,T];L^{2+\sigma}(\mathbb{T})) \times \mathcal{W}_{\mathrm{T}}$ denote the unique solution to the Cauchy problem* (3.80)*. Then, the function $u_\lambda := |a_\lambda|^2$ with $u_\lambda \in C^0([0,T];L^{1+\sigma/2}(\mathbb{T}))$ is a distributional solution to* (3.83) *in the sense that u_λ satisfies the following equation for all $\psi \in C_c^\infty((0,T)\times\mathbb{T})$*

$$\int_0^t \int_{\mathbb{T}} u_\lambda \cdot \big(\partial_t\psi + \partial_x\psi\big)\,dx\,ds = 2\alpha \int_0^t \int_{\mathbb{T}} \big(\chi_{\Omega_{\mathrm{act}}}\,w_\lambda\,u_\lambda\big)\cdot\psi\,dx\,ds. \tag{3.84}$$

Proof. We consider the time-shifted functions

$$\widetilde{u}_\lambda(x,t) := u_\lambda(x+t,t), \qquad \widetilde{a}_\lambda(x,t) := a_\lambda(x+t,t), \tag{3.85a}$$
$$\widetilde{w}_\lambda(x,t) := w_\lambda(x+t,t), \qquad \widetilde{\chi}_{\Omega_{\mathrm{act}}}(x,t) := \chi_{\Omega_{\mathrm{act}}}(x+t,t), \tag{3.85b}$$

and recall that from the proof of Lemma 3.3.3 we have

$$\frac{d}{dt}\widetilde{a}_\lambda(x,t) = -\alpha\,\widetilde{\chi}_{\Omega_{\rm act}}(x,t)\,\widetilde{w}_\lambda(x,t)\,\widetilde{a}_\lambda(x,t). \tag{3.86}$$

Similar as in the proof of Lemma 3.3.3 we get that for every $\psi \in C_c^\infty((0,T)\times\mathbb{T})$ it holds

$$\begin{aligned}\int_0^t\int_{\mathbb{T}} u_\lambda(x,s)\cdot(\partial_t+\partial_x)\,\psi(x,s)\,dx\,ds &= \int_0^t\int_{\mathbb{T}} \widetilde{u}_\lambda(x,s)\cdot\frac{d}{ds}\psi(x+s,s)\,dx\,ds\\ &= -2\int_0^t\int_{\mathbb{T}}\widetilde{a}_\lambda(x,s)\,\frac{d}{ds}\widetilde{a}_\lambda(x,s)\cdot\psi(x+s,s)\,dx\,ds\\ &= 2\alpha\int_0^t\int_{\mathbb{T}}\widetilde{\chi}_{\Omega_{\rm act}}(x,s)\,\big|\widetilde{a}_\lambda(x,s)\big|^2\,\widetilde{w}_\lambda(x,s)\cdot\psi(x+s,s)\,dx\,ds. \end{aligned}\tag{3.87}$$

Recalling (3.85) and $u_\lambda := |a_\lambda|^2$, this proves the lemma. □

Recalling the definition $Q_T := [0,T]\times\mathbb{T}$, we can infer from the bounds from Lemma 3.4.1 that there exist some subsequence $\{(a_\lambda,w_\lambda)\}_{\lambda\geq 1}$ and some pair $(a_\infty,w_\infty)\in L^2(Q_T)\times(L^2\cap L^\infty)(Q_T)$ such that

$$a_\lambda \longrightarrow a_\infty \qquad \text{weakly in } L^2(Q_T), \tag{3.88}$$
$$w_\lambda \longrightarrow w_\infty \qquad \text{weakly in } L^2(Q_T) \text{ and weakly-}* \text{ in } L^\infty(Q_T). \tag{3.89}$$

Furthermore, possibly after extracting another subsequence, we can also infer the existence of some $a_{\rm sq}\in L^{1+\sigma/2}(Q_T)$ such that

$$|a_\lambda|^2 \longrightarrow a_{\rm sq} \qquad \text{weakly in } L^{1+\sigma/2}(Q_T). \tag{3.90}$$

We stress that it is not clear whether $a_{\rm sq} = |a_\infty|^2$ holds. For the rest of this chapter we fix the properties of the subsequence under consideration.

Declaration 3.4.3. *In the following, let $\{(a_\lambda,w_\lambda)\}_{\lambda\geq1}$ denote a subsequence of the sequence of solutions to the Cauchy problems* (3.80) *with $\lambda\geq 1$ for which the weak convergences* (3.88)–(3.90) *hold. Moreover, let*

$$(a_\infty,w_\infty)\in L^2(Q_T)\times(L^2\cap L^\infty)(Q_T) \qquad \textit{and} \qquad a_{sq}\in L^{1+\sigma/2}(Q_T) \tag{3.91}$$

denote the corresponding weak limits. We recall that for all $\lambda\geq1$ we have

$$a_\lambda \in L^\infty((0,T)\times\mathbb{T})\cap C^0([0,T];L^p(\mathbb{T})) \qquad \forall\, p\in[2,\infty), \tag{3.92}$$
$$w_\lambda \in \mathcal{W}_{\rm T}\cap C^0([0,T];L^p(\mathbb{T})) \qquad \forall\, p\in[1,\infty). \tag{3.93}$$

Moreover, we introduce the following constants noting that $\|u\|^2_{L^{2+\sigma}} = \||u|^2\|_{L^{1+\sigma/2}}$ holds

$$C_w := \sup_{\lambda\geq1}\big\|w_\lambda\big\|_{L^2(Q_T)}, \qquad C_a := \sup_{\lambda\geq1}\big\|a_\lambda\big\|_{L^2(Q_T)}, \qquad C_a^\sigma := \sup_{\lambda\geq1}\big\|a_\lambda\big\|^2_{L^{2+\sigma}(Q_T)}. \tag{3.94}$$

The rest of this section is now devoted to the proof of the following proposition, which implies Theorem 3.2.2.

Proposition 3.4.4. *For the weak limit (a_∞, w_∞) of the subsequence from Declaration 3.4.3 we have*

$$(a_\infty, w_\infty) \in C^0([0,T];(L^2 \cap L^{2+\sigma})(\mathbb{T})) \times \big(C^0([0,T];L^p(\mathbb{T})) \cap L^\infty((0,T)\times\mathbb{T})\big) \quad (3.95)$$

and the subsequence from Declaration 3.4.3 satisfies

$$(a_\lambda, w_\lambda) \longrightarrow (a_\infty, w_\infty) \qquad \text{strongly in } C^0([0,T];L^2(\mathbb{T})). \quad (3.96)$$

The pair (a_∞, w_∞) is a weak solution to the Cauchy problem for (3.33) *with initial data (a_0, w_0) in the sense of Definition 3.1.3.*

For the proof of this proposition we will proceed as follows. First, in Section 3.4.2, we will prove that $\{w_\lambda\}_{\lambda\geq 1}$ is a Cauchy sequence in the space $L^2(Q_T)$. Using this result, we can prove the strong convergence $a_\lambda \longrightarrow a_\infty$ in $C^0([0,T];L^2(\mathbb{T}))$ and conclude. This is the content of in Section 3.4.3.

3.4.2. Strong Convergence of the Inversion

We begin with the crucial step and show the following strong convergence result.

Lemma 3.4.5. *The sequence $\{w_\lambda\}_{\lambda\geq 1}$ from Declaration 3.4.3 is a Cauchy sequence with limit w_∞ in the space $L^2(Q_T)$.*

Proof. Testing the difference of the equations for w_λ and w_μ with $(w_\lambda - w_\mu)$ yields the pointwise estimate

$$\begin{aligned}
&\partial_t(w_\lambda - w_\mu)(w_\lambda - w_\mu)\\
&\quad = \Big(\alpha\,\chi_{\Omega_{\text{act}}}\big(h_\delta(w_\lambda)|a_\lambda|^2 - h_\delta(w_\mu)|a_\mu|^2\big) - \kappa(w_\lambda - w_\mu)\Big)(w_\lambda - w_\mu)\\
&\quad = \Big(\alpha\,\chi_{\Omega_{\text{act}}}\big(\big(h_\delta(w_\lambda)\big(|a_\lambda|^2 - a_{\text{sq}}\big) - h_\delta(w_\mu)\big(|a_\mu|^2 - a_{\text{sq}}\big)\big)\\
&\qquad + \alpha\,\chi_{\Omega_{\text{act}}}\big(h_\delta(w_\lambda) - h_\delta(w_\mu)\big)a_{\text{sq}} - \kappa(w_\lambda - w_\mu)\Big)(w_\lambda - w_\mu)\\
&\quad \leq L_{h_\delta}|\alpha|\,\chi_{\Omega_{\text{act}}}\,\big|a_{\text{sq}}\big|\,|w_\lambda - w_\mu|^2 +\\
&\qquad \alpha\,\chi_{\Omega_{\text{act}}}\Big(h_\delta(w_\lambda)\big(|a_\lambda|^2 - a_{\text{sq}}\big) - h_\delta(w_\mu)\big(|a_\mu|^2 - a_{\text{sq}}\big)\Big)(w_\lambda - w_\mu). \qquad (3.97)
\end{aligned}$$

The term $L_{h_\delta}|\alpha|\,a_{\text{sq}}\,|w_\lambda - w_\mu|^2$ is difficult to handle, because we cannot make a statement about its behavior in the limit. In order to eliminate this term, we introduce the weight $\exp(-b)$ where the positive, measurable and almost everywhere finite function b is defined in terms of the weak limit a_{sq} by

$$b(x,t) := \int_0^t L_{h_\delta}|\alpha|\,a_{\text{sq}}(x,s)\,ds. \quad (3.98)$$

We obviously have $\partial_t b(x,t) = L_{h_\delta}|\alpha|\, a_{\text{sq}}(x,t)$ and

$$(x,t) \longmapsto \exp\big(-b(x,t)\big) \quad \in L^\infty(Q_T). \tag{3.99}$$

With the weight $\exp(-b)$, we get the following pointwise estimate

$$\begin{aligned}
\frac{1}{2}\partial_t\big(e^{-2b}|w_\lambda - w_\mu|^2\big) &= -\partial_t b\, e^{-2b}|w_\lambda - w_\mu|^2 + \frac{1}{2}e^{-2b}\partial_t|w_\lambda - w_\mu|^2 \\
&\le - L_{h_\delta}|\alpha|\,\chi_{\Omega_{\text{act}}}\, a_{\text{sq}}\, e^{-2b}|w_\lambda - w_\mu|^2 + e^{-2b}L_{h_\delta}|\alpha|\, a_{\text{sq}}\,|w_\lambda - w_\mu|^2 \\
&\quad + e^{-2b}\alpha\,\chi_{\Omega_{\text{act}}}\Big(h_\delta(w_\lambda)\big(|a_\lambda|^2 - a_{\text{sq}}\big) - h_\delta(w_\mu)\big(|a_\mu|^2 - a_{\text{sq}}\big)\Big)(w_\lambda - w_\mu) \\
&= e^{-2b}\alpha\,\chi_{\Omega_{\text{act}}}\Big(h_\delta(w_\lambda)\big(|a_\lambda|^2 - a_{\text{sq}}\big) - h_\delta(w_\mu)\big(|a_\mu|^2 - a_{\text{sq}}\big)\Big)(w_\lambda - w_\mu).
\end{aligned} \tag{3.100}$$

Integrating over $\Omega_t := (0,t) \times \Omega_{\text{act}}$ with some arbitrary $t \in (0,T)$ yields

$$\begin{aligned}
&\frac{1}{2}\big\|e^{-b(t)}(w_\lambda - w_\mu)(t)\big\|^2_{L^2(\Omega_{\text{act}})} \\
&\le \int_{\Omega_t} e^{-2b}\alpha\Big(h_\delta(w_\lambda)\big(|a_\lambda|^2 - a_{\text{sq}}\big) - h_\delta(w_\mu)\big(|a_\mu|^2 - a_{\text{sq}}\big)\Big)(w_\lambda - w_\mu)\,dx\,ds \\
&= \alpha\int_{\Omega_t} e^{-2b}h_\delta(w_\lambda)\big(|a_\lambda|^2 - a_{\text{sq}}\big)(w_\lambda - w_\mu)\,dx\,ds && (3.101) \\
&\quad - \alpha\int_{\Omega_t} e^{-2b}h_\delta(w_\mu)\big(|a_\mu|^2 - a_{\text{sq}}\big)(w_\lambda - w_\mu)\,dx\,ds. && (3.102)
\end{aligned}$$

Interchanging λ and μ in (3.102) yields (3.101). Therefore, in order to show that the sum (3.101)+(3.102) tends to zero in the limit, it suffices to show the convergence

$$\forall\, t \in [0,T]: \qquad \lim_{\lambda,\mu\to\infty} \alpha\int_{\Omega_t} e^{-2b}h_\delta(w_\lambda)\big(|a_\lambda|^2 - a_{\text{sq}}\big)(w_\lambda - w_\mu)\,dx\,ds = 0. \tag{3.103}$$

Next, we involve a version of the div-curl lemma (see Theorem A.6.3) to show this convergence. To this end, we define the vector fields

$$u_\lambda := \big(|a_\lambda|^2 - a_{\text{sq}}, |a_\lambda|^2 - a_{\text{sq}}\big), \qquad v_{\lambda,\mu} := \big(0,\, h_\delta(w_\lambda)(w_\lambda - w_\mu)\big) \tag{3.104}$$

and the corresponding operators div and curl by

$$\operatorname{div} u := \partial_t u_1 + \partial_x u_2, \qquad \operatorname{curl} v := \begin{pmatrix} 0 & \partial_x v_1 - \partial_t v_2 \\ \partial_t v_2 - \partial_x v_1 & 0 \end{pmatrix}. \tag{3.105}$$

In a similar way, this idea has also been used in [Mie99]. We note that the following argumentation is valid for all $t \in [0,T]$ and set

$$p := 1 + \sigma/2 \qquad \text{and} \qquad q := (2+\sigma)/\sigma. \tag{3.106}$$

It is easy to check that p and q are conjugate exponents, i.e. they satisfy $p^{-1} + q^{-1} = 1$. Due to the bounds (3.81a) on $\{w_\lambda\}_{\lambda \geq 1}$ and the upper bound $h_\delta(\cdot) \leq 1/2$, we may infer

$$\begin{aligned}\forall\, \lambda, \mu \geq 1: \qquad \big\|v_{\lambda,\mu}\big\|_{L^q(\Omega_t)} &= \big\|h_\delta(w_\lambda)\,(w_\lambda - w_\mu)\big\|_{L^q(\Omega_t)} \\ &\leq \frac{1}{2}\big(\big\|w_\lambda\big\|_{L^\infty(\Omega_t)} + \big\|w_\mu\big\|_{L^\infty(\Omega_t)}\big)\, |\Omega_t|^{1/q} \leq |\Omega_t|^{1/q}. \quad (3.107)\end{aligned}$$

Thus, the family $\{v_{\lambda,\mu}\}_{\lambda,\mu \geq 1}$ is bounded in the space $L^q(\Omega_t)$. Therefore, there exist $v_\infty \in L^q(\Omega_t)$ and a subsequence $\{v_{\lambda_k,\mu_k}\}_{k\in\mathbb{N}} \subset \{v_{\lambda,\mu}\}_{\lambda,\mu\geq 1}$ such that $v_{\lambda_k,\mu_k} \longrightarrow v_\infty$ weakly in $L^q(\Omega_t)$ as $k \to \infty$. Recalling the weak convergence of $\{|a_\lambda|^2\}_{\lambda\geq 1}$ from (3.90), we conclude that the following convergences hold as $k \to \infty$, resp. $\lambda \to \infty$

$$u_\lambda \longrightarrow u_\infty = 0 \quad \text{weakly in } L^p(\Omega_t), \qquad v_{\lambda_k,\mu_k} \longrightarrow v_\infty \quad \text{weakly in } L^q(\Omega_t). \qquad (3.108)$$

The function $(x,t) \longmapsto e^{-2b(x,t)}$ is from the space $L^\infty(\Omega_t) \cong \big(L^1(\Omega_t)\big)^*$. Theorem A.6.3 thus yields the convergence from (3.103) for the subsequence $(u_{\lambda_k}, v_{\lambda_k,\mu_k})_{k\in\mathbb{N}}$ if the following holds

$$\operatorname{div} u_\lambda \longrightarrow \operatorname{div} u_\infty \quad \text{strongly in } W^{-1,1}(\Omega_t), \qquad (3.109)$$

$$\operatorname{curl} v_{\lambda_k,\mu_k} \longrightarrow \operatorname{curl} v_\infty \quad \text{strongly in } W^{-1,1}(\Omega_t), \qquad (3.110)$$

$$\text{the set} \quad \big\{u_\lambda \cdot v_{\lambda,\mu}\big\}_{\lambda,\mu\geq 1} \subset L^1(\Omega_t) \quad \text{is equi-integrable.} \qquad (3.111)$$

From the Lipschitz continuity of h_δ, the $L^{1+\sigma/2}$ bounds on $|a_\lambda|^2$ and the L^∞ bounds on w_λ from (3.94), we get the following bound by using that w_λ is a solution to the differential equation (3.33b)

$$\begin{aligned}\sup_{\lambda,\mu\geq 1} \big\|\operatorname{curl} v_{\lambda,\mu}\big\|_{L^p(\Omega_t)} &= 2 \sup_{\lambda,\mu\geq 1} \big\|\partial_t\big(h_\delta(w_\lambda)(w_\lambda - w_\mu)\big)\big\|_{L^p(\Omega_t)} \\ &\leq 2 \sup_{\lambda,\mu\geq 1} \Big(\big\|L_{h_\delta}|\partial_t w_\lambda|(w_\lambda - w_\mu)\big\|_{L^p(\Omega_t)} + \tfrac{1}{2}\big\|\partial_t(w_\lambda - w_\mu)\big\|_{L^p(\Omega_t)}\Big) \\ &\leq 6\, L_{h_\delta} \sup_{\lambda\geq 1} \Big(\tfrac{1}{2}|\alpha|\big\||a_\lambda|^2\big\|_{L^p(\Omega_t)} + \kappa\big\|w_\lambda\big\|_{L^p(\Omega_t)}\Big) \\ &\leq C(L_{h_\delta}, \alpha, \kappa) \cdot \big(C_a^\sigma + \big|\Omega_t\big|^{1/p}\big) \leq C(L_{h_\delta}, C_a^\sigma, C_w, \alpha, \kappa). \quad (3.112)\end{aligned}$$

Similarly, due to the L^∞ bounds on $w_{\lambda,\mu}$, the $L^{1+\sigma/2}$ bounds on $|a_\lambda|^2$ and taking into account Lemma 3.4.2, we get the bound

$$\begin{aligned}\sup_{\lambda\geq 1} \big\|\operatorname{div} u_\lambda\big\|_{L^p(\Omega_t)} &= \sup_{\lambda\geq 1} \big\|(\partial_t + \partial_x)|a_\lambda|^2\big\|_{L^p(\Omega_t)} \\ &\leq 2\,|\alpha| \sup_{\lambda\geq 1} \big\|\chi_{\Omega_{\text{act}}} w_\lambda\, |a_\lambda|^2\big\|_{L^p(\Omega_t)} \leq 2\,|\alpha|\, C_a^\sigma. \qquad (3.113)\end{aligned}$$

As shown in Lemma A.6.4, the embedding $L^p(\Omega_t) \hookrightarrow W^{-1,1}(\Omega_t)$ is compact since Ω_t is open and bounded. Therefore, from the above bounds (3.112)–(3.113), we may infer the following strong convergences[8]

$$\operatorname{div} u_\lambda \longrightarrow \operatorname{div} u_\infty \quad \text{in } W^{-1,1}(\Omega_t), \qquad \operatorname{curl} v_{\lambda_k,\mu_k} \longrightarrow \operatorname{curl} v_\infty \quad \text{in } W^{-1,1}(\Omega_t). \tag{3.114}$$

We are left with showing that the set $\{u_\lambda \cdot v_{\lambda,\mu}\}_{\lambda,\mu\geq 1}$ is equi-integrable. To this end, we consider the estimate

$$\begin{aligned}\forall\, \lambda, \mu \geq 1: \qquad \big\|v_{\lambda,\mu}\big\|_{L^\infty(\Omega_t)} &= \big\|h_\delta(w_\lambda)\,(w_\lambda - w_\mu)\big\|_{L^\infty(\Omega_t)} \\ &\leq \frac{1}{2}\big(\big\|w_\lambda\big\|_{L^\infty(\Omega_t)} + \big\|w_\mu\big\|_{L^\infty(\Omega_t)}\big) = 1.\end{aligned} \tag{3.115}$$

This means that for all $\lambda, \mu \geq 1$, we have $v_{\lambda,\mu} \in L^\infty(\Omega_t)$. In particular, this implies that for all $\lambda, \mu \geq 1$, we have $u_\lambda \cdot v_{\lambda,\mu} \in L^{1+\sigma/2}(\Omega_t)$. Involving the estimate (3.94) yields

$$\sup_{\lambda,\mu\geq 1} \big\|u_\lambda \cdot v_{\lambda,\mu}\big\|_{L^{1+\sigma/2}(\Omega_t)} \leq \sup_{\lambda,\mu\geq 1} \big\|u_\lambda\big\|_{L^{1+\sigma/2}(\Omega_t)} \big\|v_{\lambda,\mu}\big\|_{L^\infty(\Omega_t)} \leq 2\, C_a^\sigma. \tag{3.116}$$

Thus, the set $\{u_\lambda \cdot v_{\lambda,\mu}\}_{\lambda,\mu\geq 1}$ is bounded in the space $L^{1+\sigma/2}(\Omega_t)$. The De La Vallée–Poussin theorem (see Theorem A.1.18) yields the equi-integrability of the set $\{u_\lambda \cdot v_{\lambda,\mu}\}_{\lambda,\mu\geq 1}$. Hence, we have shown the convergence from (3.103) for the subsequence $(u_{\lambda_k}, v_{\lambda_k,\mu_k})_{k\in\mathbb{N}}$, i.e.

$$\forall\, t \in [0,T]: \qquad \lim_{k\to\infty} \alpha \int_{\Omega_t} e^{-2b} h_\delta(w_{\lambda_k})\big(|a_{\lambda_k}|^2 - a_{\mathrm{sq}}\big)(w_{\lambda_k} - w_{\mu_k})\,dx\,ds = 0. \tag{3.117}$$

In fact, the convergence (3.117) holds for the full subsequence from Declaration 3.4.3, i.e. it holds (3.103). To see this, note that every subsequence of the family $\{v_{\lambda,\mu}\}_{\lambda,\mu\geq 1}$ has a weakly convergent subsequence. The limits may differ, but the above convergence (3.117) does not depend on the limit. Therefore, the convergence (3.117) holds for every subsequence from the family $\{v_{\lambda,\mu}\}_{\lambda,\mu\geq 1}$, i.e. it holds (3.103) for the full subsequence from Declaration 3.4.3. Moreover, from (3.103) we get the convergence

$$\forall\, t \in [0,T]: \qquad \big\|e^{-b(t)}(w_\lambda - w_\mu)(t)\big\|_{L^2(\Omega_{\mathrm{act}})} \longrightarrow 0, \qquad \lambda, \mu \to \infty. \tag{3.118}$$

In particular, this implies

$$\big\|e^{-b}(w_\lambda - w_\mu)\big\|_{L^2(Q_T)} \longrightarrow 0, \qquad \lambda, \mu \to \infty. \tag{3.119}$$

Thus, the subsequence $\{w_\lambda\}_{\lambda\geq 1}$ from Declaration 3.4.3 is a Cauchy sequence in the space $L^2(Q_T, e^{-b}dx\,dt)$. Due to the weak convergence $w_\mu \longrightarrow w_\infty$ in $L^2(Q_T)$ and the weak lower semi-continuity of the norm we get

$$\big\|e^{-b}(w_\lambda - w_\infty)\big\|_{L^2(Q_T)} \longrightarrow 0 \qquad \text{as} \qquad \lambda \to \infty. \tag{3.120}$$

[8]It is not necessary to extract subsequences, because the uniqueness of the limits u_∞, v_∞ yields that all subsequences converge to the same limits and the convergences actually hold for the full sequences.

We end the proof by showing the strong convergence $w_\lambda \longrightarrow w_\infty$ in $L^2(Q_T)$. To this end, we define

$$W_\lambda := \int_{Q_T} \left|w_\lambda - w_\infty\right|^2 dx\, dt, \qquad \widetilde{w}_\lambda(x,t) := \left|w_\lambda(x,t) - w_\infty(x,t)\right|^2. \tag{3.121}$$

On the one hand, due to the L^2-bounds on w_λ from (3.81a) we see that $\{W_\lambda\}_{\lambda\geq 1} \subset \mathbb{R}$ is a bounded set. Therefore, we may infer the existence of some subsequence $\{W_{\lambda_k}\}_{k\in\mathbb{N}} \subset \{W_\lambda\}_{\lambda\geq 1}$ and some $W_\infty \geq 0$ with

$$W_\infty := \limsup_{\lambda\to\infty} W_\lambda = \lim_{k\to\infty} W_{\lambda_k}. \tag{3.122}$$

On the other hand, since b is positive and finite almost everywhere, it holds $e^{-b} \neq 0$ a.e. in $(0,T)\times\mathbb{T}$. Thus, due to the convergence from (3.120) and due to Weyl's theorem, we may extract a subsequence $\{w_{\lambda_k}\}_{k\in\mathbb{N}} \subset \{w_\lambda\}_{\lambda\geq 1}$ which converges pointwise a.e. in Q_T to w_∞ w.r.t. the measure $dx\,dt$ as well as $e^{-b}\,dx\,dt$. In particular, this implies that we may also extract a sub-subsequence $\{\widetilde{w}_{\lambda_{k_j}}\}_{j\in\mathbb{N}} \subset \{\widetilde{w}_{\lambda_k}\}_{k\in\mathbb{N}}$, that converges pointwise a.e. in Q_T to zero as $j\to\infty$. Thanks to the L^∞-bounds on w_λ from (3.81a), we may infer with Lebesgue's convergence theorem the strong convergence $\widetilde{w}_{\lambda_{k_j}} \longrightarrow 0$ in $L^1(Q_T)$. Due to the uniqueness of the limit, this implies $W_\infty = 0$ as well as the convergence

$$w_\lambda \longrightarrow w_\infty \qquad \text{strongly in } L^2(Q_T) \tag{3.123}$$

for the full sequence. This proves Lemma 3.4.5. □

3.4.3. Strong Convergence of the Amplitude and Conclusion

Next, using the strong convergence result from the preceding Lemma 3.4.5, we show the strong convergence of the sequence $\{a_\lambda\}_{\lambda\geq 1}$ from Declaration 3.4.3. Subsequently, we conclude.

Lemma 3.4.6. *The subsequence $\{a_\lambda\}_{\lambda\geq 1}$ from Declaration 3.4.3 strongly converges to a_∞ in the space $C^0([0,T];(L^2\cap L^{2+\sigma})(\mathbb{T}))$.*

Proof. Recalling the definition of $\xi^t_\tau := x+\tau-t$ and insertion of both, the initial datum a_0 and the function w_∞, into the solution formula (3.51) for the amplitude a yields the well defined function

$$a^\infty(x,t) := a_0(x-t)\cdot \exp\Big(-\alpha \int_0^t w_\infty(\xi^t_\tau,\tau)\,d\tau\Big) \in C^0([0,T];(L^2\cap L^{2+\sigma})(\mathbb{T})). \tag{3.124}$$

Next, for arbitrary $t\in(0,T)$ we consider simultaneously the difference of $a^\infty(t)$ and $a_\lambda(t)$ in the $L^p(\mathbb{T})$-norms for $p=2$ and $p=(2+\sigma)$. We aim to show that this difference tends to zero in the limit uniformly in t. From Lemma A.1.13 we get that for both $p=2$

and $p = (2+\sigma)$ the following estimate holds for all $t \in (0, T)$

$$
\begin{aligned}
&\big\|a^\infty(t) - a_\lambda(t)\big\|_{L^p(\mathbb{T})} = \\
&\Big\|a_0(x-t)\,\exp\Big(-\alpha\int_0^t w_\infty(\xi_\tau^t,\tau)\,d\tau\Big) - a_0^\lambda(x-t)\,\exp\Big(-\alpha\int_0^t w_\lambda(\xi_\tau^t,\tau)\,d\tau\Big)\Big\|_{L^p(\mathbb{T})} \\
&\leq \Big\|\big(a_0^\lambda(x-t) - a_0(x-t)\big)\,\exp\Big(|\alpha|\int_0^t 1\,d\tau\Big)\Big\|_{L^p(\mathbb{T})} \\
&\quad + \Big\|a_0(x-t)\,\Big|\exp\Big(-\alpha\int_0^t w_\lambda(\xi_\tau^t,\tau)\,d\tau\Big) - \exp\Big(-\alpha\int_0^t w_\infty(\xi_\tau^t,\tau)\,d\tau\Big)\Big|\,\Big\|_{L^p(\mathbb{T})} \\
&\leq \big\|a_0^\lambda - a_0\big\|_{L^p(\mathbb{T})}\cdot\exp\big(|\alpha|T\big) && (3.125) \\
&\quad + \Big\|a_0(x-t)\cdot C_\alpha(T)\int_0^t |w_\lambda(\xi_\tau^t,\tau) - w_\infty(\xi_\tau^t,\tau)|\,d\tau\Big\|_{L^p(\mathbb{T})}, && (3.126)
\end{aligned}
$$

with $C_\alpha(T) := |\alpha|\exp(|\alpha|T)$. In the limit $\lambda \to \infty$, the first summand (3.125) vanishes due to the strong convergence (3.79). With $\xi_\tau^t := x + \tau - t$, the term (3.126) is equal to

$$
W_\lambda(t) := C_\alpha(T)\cdot\Big\||a_0(x)|\int_0^t \big|w_\infty(x+\tau,\tau) - w_\lambda(x+\tau,\tau)\big|\,d\tau\Big\|_{L^p(\mathbb{T})}. \tag{3.127}
$$

Obviously, we have $\max_{t\in[0,T]} W_\lambda(t) = W_\lambda(T)$. Next, we involve Lemma A.1.3. We have $a_0 \in (L^2 \cap L^{2+\sigma})(\mathbb{T})$ and due to (3.81a), the sequence defined by

$$
f_\lambda(x) := \int_0^T \big|w_\infty(x+\tau,\tau) - w_\lambda(x+\tau,\tau)\big|\,d\tau \tag{3.128}
$$

satisfies $\sup_{\lambda\geq 1}\|f_\lambda\|_{L^\infty(\mathbb{T})} \leq C$ with $C \leq 2T$. Therefore, in order to show the convergence $W_\lambda(T) \longrightarrow 0$, in view of Lemma A.1.3, it suffices to show the convergence $f_\lambda \longrightarrow 0$ in the space $L^2(\mathbb{T})$. To show this convergence, we consider

$$
\begin{aligned}
&\int_{\mathbb{T}} |f_\lambda(x)|^2\,dx = \int_{\mathbb{T}}\Big|\int_0^T \big|w_\lambda(x+\tau,\tau) - w_\infty(x+\tau,\tau)\big|\,d\tau\Big|^2\,dx \\
&= \int_{\mathbb{T}}\Big|\int_0^T \big|w_\lambda(x+\tau,\tau) - w_\infty(x+\tau,\tau)\big|\,d\tau\Big|\cdot\Big|\int_0^T \big|w_\lambda(x+\tau,\tau) - w_\infty(x+\tau,\tau)\big|\,d\tau\Big|\,dx \\
&\leq \int_{\mathbb{T}}\Big|\int_0^T \big|w_\lambda(x+\tau,\tau) - w_\infty(x+\tau,\tau)\big|\,d\tau\Big|\cdot\Big(\|w_\lambda\|_{L^\infty(Q_T)} + \|w_\infty\|_{L^\infty(Q_T)}\Big)\cdot T\,dx. \\
&\leq 2\,T\int_{\Omega_{\mathrm{act}}}\int_0^T \big|w_\lambda(x,\tau) - w_\infty(x,\tau)\big|\,d\tau\,dx.
\end{aligned}
$$

Fubini's theorem and Hölder's estimate yield

$$
\leq C(\Omega_{\mathrm{act}},T)\cdot\big\|w_\lambda - w_\infty\big\|_{L^2(\Omega_T)} = C(\Omega_{\mathrm{act}},T)\cdot\big\|w_\lambda - w_\infty\big\|_{L^2(Q_T)}. \tag{3.129}
$$

The strong convergence of the inversion from (3.123) shows $f_\lambda \longrightarrow 0$ in $L^2(\mathbb{T})$. Therefore, Lemma A.1.3 yields the convergence $W_\lambda(T) \longrightarrow 0$. Thus, we have shown the convergence

$$\left\|a^\infty - a_\lambda\right\|_{C^0([0,T];L^p(\mathbb{T}))} \longrightarrow 0 \quad \text{as } \lambda \to \infty \quad \text{for } p \in \{2, (2+\sigma)\}. \tag{3.130}$$

With the usual identifications we may infer $a^\infty = a_\infty$ due to the uniqueness of the limit. This proves Lemma 3.4.6. □

Conclusion

It remains to show that (a_∞, w_∞) is in fact a weak solution to Problem 3.1.5 in the sense of Definition 3.1.3. To prove that (a_∞, w_∞) satisfies (3.11), we take the limits[9] in the equation

$$\int_{\mathbb{T}} \varphi(x)\big(a_\lambda(x,t) - a_0^\lambda(x)\big)\, dx = \int_0^t \int_{\mathbb{T}} \big(\partial_x\varphi(x)\, a_\lambda(x,t) + \alpha\, a_\lambda(x,t)\, w_\lambda(x,t)\, \varphi(x)\big)\, dx\, dt$$

with arbitrary $\varphi \in W^{1,2}(\mathbb{T})$. To see that w_∞ solves the ODE, we define the function

$$w^\infty(x,t) := \int_0^t \alpha\, \chi_{\Omega_{\text{act}}}(x)\, h_\delta\big(w_\infty(x,s)\big)\, |a_\infty(x,s)|^2 - \kappa\, w_\infty(x,s)\, ds. \tag{3.131}$$

Next, we consider the following difference for arbitrary $t \in (0,T)$

$$\begin{aligned}
&\int_{\mathbb{T}} \big|w_\lambda(t) - w^\infty(t)\big|\, dx = \int_{\mathbb{T}} \Big|w_\lambda(t) - \int_0^t \alpha\, \chi_{\Omega_{\text{act}}} \big(h_\delta(w_\infty)\, |a_\infty|^2 - \kappa w_\infty\big)\, ds\Big|\, dx \\
&= \int_{\mathbb{T}} \Big| \int_0^t \alpha\, \chi_{\Omega_{\text{act}}} \big(h_\delta(w_\infty)\, |a_\infty|^2 - h_\delta(w_\lambda)\, |a_\lambda|^2\big) - \kappa(w_\infty - w_\lambda)\, ds\Big|\, dx \\
&\le \int_0^T \Big(\int_{\Omega_{\text{act}}} |\alpha|\, \big|h_\delta(w_\infty)\, |a_\infty|^2 - h_\delta(w_\lambda)\, |a_\lambda|^2\big|\, dx + \kappa \int_{\Omega_{\text{act}}} |w_\infty - w_\lambda|\, dx\Big)\, ds.
\end{aligned} \tag{3.132}$$

The second summand in the last line tends to zero in the limit $\lambda \to \infty$ due to the strong convergence $w_\lambda \longrightarrow w_\infty$ in $L^2(Q_T)$ from Lemma 3.4.5. For the first summand, we get the following estimate from the Lipschitz continuity of h_δ and the bound $h_\delta(\cdot) \le 1/2$

$$\begin{aligned}
&|\alpha| \int_0^T \int_{\Omega_{\text{act}}} \big|h_\delta(w_\infty)\, |a_\infty|^2 - h_\delta(w_\lambda)\, |a_\lambda|^2\big|\, dx\, ds = \\
&|\alpha| \int_0^T \int_{\Omega_{\text{act}}} \big|h_\delta(w_\infty) - h_\delta(w_\lambda)\big|\, |a_\infty|^2 + h_\delta(w_\lambda)\, \big||a_\infty|^2 - |a_\lambda|^2\big|\, dx\, ds \\
&\le |\alpha| L_{h_\delta} \int_0^T \int_{\Omega_{\text{act}}} \big|w_\infty - w_\lambda\big|\, |a_\infty|^2\, dx\, ds + \frac{|\alpha|}{2} \int_0^T \int_{\Omega_{\text{act}}} \big||a_\infty|^2 - |a_\lambda|^2\big|\, dx\, ds.
\end{aligned} \tag{3.133}$$

[9]Clearly, for all $\varphi \in W^{1,2}(\mathbb{T})$ we have $\varphi\, a_\lambda \longrightarrow \varphi\, a_\infty$ strongly in $L^1(Q_T)$. Therefore, the weak-$*$ convergence $w_\lambda \longrightarrow w_\infty$ in $L^\infty(Q_T)$ from (3.89) yields $\int_{Q_T} a_\lambda\, w_\lambda\, \varphi\, dx\, dt \longrightarrow \int_{Q_T} a_\infty\, w_\infty\, \varphi\, dx\, dt$.

Due to the strong convergence $a_\lambda \longrightarrow a_\infty$ in $L^2(Q_T)$ from Lemma 3.4.6, the second summand tends to zero in the limit[10]. Involving Lemma A.1.3, we may infer that also the first summand tends to zero as $\lambda \to \infty$. Obviously, the convergence is uniform in t. Thus, we have shown the convergence

$$\|w_\lambda - w^\infty\|_{C^0([0,T];L^1(\mathbb{T}))} \longrightarrow 0. \tag{3.134}$$

With the usual identification, the uniqueness of the limit implies $w_\infty = w^\infty$. Due to our assumption (3.89) it holds $w_\infty \in L^\infty((0,T)\times\mathbb{T})$. Hence, the function w_∞ is a weak solution to the Cauchy problem for (3.33b) in the sense of Definition 3.1.3. In particular, taking into account Proposition 3.1.4, it holds $w_\infty \in C^0([0,T];L^p(\mathbb{T}))$ for all $p \in [1,\infty)$. The convergence $\|w_\lambda - w_\infty\|_{C^0([0,T];L^2(\mathbb{T}))} \longrightarrow 0$ is immediate due to the bound (3.16) on w_∞, due to the bounds (3.81a) on $\{w_\lambda\}_{\lambda\geq 1}$ and due to the convergence from (3.134). This ends the proof of Proposition 3.4.4.

3.5. An Existence and Uniqueness Proof for the Non-Lipschitzian Case

In Section 3.3 we showed existence and uniqueness for the solution to the Cauchy problem of the regularized system

$$\partial_t a(x,t) = -\partial_x a(x,t) - \alpha\, \chi_{\Omega_{\text{act}}}(x)\, w(x,t)\, a(x,t) \tag{3.33a}$$

$$\partial_t w(x,t) = \alpha\, \chi_{\Omega_{\text{act}}}(x)\, h_\delta(w)\, |a(x,t)|^2 - \kappa\, w(x,t) \tag{3.33b}$$

with initial data $(a_0, w_0) \in (L^2 \cap L^\infty)(\mathbb{T}) \times \mathcal{W}_0$. In this section, we show existence and uniqueness for the original problem

$$\partial_t a(x,t) = -\partial_x a(x,t) - \alpha\, \chi_{\Omega_{\text{act}}}(x)\, w(x,t)\, a(x,t) \tag{3.9a}$$

$$\partial_t w(x,t) = \alpha\, \chi_{\Omega_{\text{act}}}(x)\, h(w)\, |a(x,t)|^2 - \kappa\, w(x,t) \tag{3.9b}$$

with initial data $(a_0, w_0) \in (L^2 \cap L^\infty)(\mathbb{T}) \times \widehat{\mathcal{W}_0}$ where

$$\widehat{\mathcal{W}_0} := \Big\{ w_0 \in \mathrm{L}^\infty(\mathbb{T}) : \ \|w_0\|_{\mathrm{L}^\infty(\mathbb{T})} < 1,\ \|w_0\|_{L^\infty(\mathbb{T}\setminus\Omega_{\text{act}})} = 0 \Big\}. \tag{3.137}$$

Namely, we prove Theorem 3.2.3. This theorem is a direct consequence of the following lemma.

[10] See [Alt06, Cor 1.22, p. 58].

Lemma 3.5.1. *Let arbitrary initial data* $w_0 \in \widehat{\mathcal{W}_0}$ *and arbitrary data* $\alpha \in \mathbb{R}$, $\kappa > 0$, $a \in \mathrm{L}^\infty((0,\mathrm{T}) \times \mathbb{T})$ *as well as an arbitrary time* $T > 0$ *be given. Then, every solution* w *to the Cauchy problem*

$$\partial_t w(x,t) = \alpha\, \chi_{\Omega_{\text{act}}}(x)\, h(w)\, |a(x,t)|^2 - \kappa\, w(x,t) \qquad \text{in } [0,T] \times \mathbb{T} \tag{3.138a}$$

$$w(0) = w_0 \qquad \text{in } \mathbb{T} \tag{3.138b}$$

admits the following a-priori estimate. There exists $\epsilon > 0$ *such that*

$$\|w\|_{\mathrm{L}^\infty((0,\mathrm{T})\times\mathbb{T})} \leq 1 - \epsilon. \tag{3.139}$$

Proof. There is nothing to be proven for $x \in \mathbb{T} \setminus \Omega_{\text{act}}$, since for $x \in \mathbb{T} \setminus \Omega_{\text{act}}$, we always have $\partial_t |w|^2 \leq 0$. Therefore, we assume $x \in \Omega_{\text{act}}$ for the rest of the proof and neglect the appearance of the characteristic function $\chi_{\Omega_{\text{act}}}$. Next, we assume $x \in \Omega_{\text{act}}$ to be a parameter and show that pointwise for a.e. $x \in \Omega_{\text{act}}$ we have $\partial_t |w|^2 < 0$ for $|w| \geq 1 - \epsilon$. This obviously implies $|w| \leq 1 - \epsilon$ for all $t \in [0,T]$. We note that it holds $\partial_t |w|^2 < 0$, iff $w\big(\alpha h(w)|a|^2 - \kappa w\big) < 0$. Since the case $\alpha = 0$ is trivial, we study the other two cases.

(i) Assume $\alpha > 0$, then the condition $w\big(\alpha h(w)|a|^2 - \kappa w\big) < 0$ is always fulfilled for $w < 0$. For $w > 0$, we have

$$w\big(\alpha h(w)|a|^2 - \kappa w\big) < 0 \quad \text{iff} \quad \frac{h(w)}{w} < \frac{\kappa}{\alpha |a|^2}. \tag{3.140}$$

For $\frac{h(w)}{w} < \frac{\kappa}{\alpha \|a\|^2_{\mathrm{L}^\infty((0,\mathrm{T})\times\mathbb{T})}}$, this condition is always fulfilled. Since we have $\frac{h(w)}{w} \to 0$ for $w \to 1$, there exists $\epsilon > 0$, s.t. for all $w \in [1-\epsilon, 1]$ it holds $\frac{h(w)}{w} < \frac{\kappa}{\alpha \|a\|^2_{\mathrm{L}^\infty((0,\mathrm{T})\times\mathbb{T})}}$.

(ii) Assume $\alpha < 0$, then the condition $w\big(\alpha h(w)|a|^2 - \kappa w\big) < 0$ is fulfilled for $w > 0$. For $w < 0$, we have

$$w\big(\alpha h(w)|a|^2 - \kappa w\big) < 0 \quad \text{iff} \quad \frac{h(w)}{w} > \frac{\kappa}{\alpha |a|^2}. \tag{3.141}$$

Note that both sides of the last inequality are negative. Taking the modulus, we can argue as before. Thus, there exists $\epsilon > 0$, s.t. for all $w \in [-1, -1+\epsilon]$ it holds $\frac{h(w)}{|w|} < \frac{\kappa}{|\alpha| \|a\|^2_{\mathrm{L}^\infty((0,\mathrm{T})\times\mathbb{T})}}$.

This proves Lemma 3.5.1. □

Due to the Lipschitz continuity of h on $[-(1-\epsilon), (1-\epsilon)]$, we can argue as in Section 3.3.3. This proves Theorem 3.2.3 with an obvious definition of the set $\widehat{\mathcal{W}_\mathrm{T}}$ given by

$$\widehat{\mathcal{W}_\mathrm{T}} := \Big\{ w \in W^{1,\infty}((0,T); L^1(\mathbb{T})) :\ \forall t \in [0,T] \text{ it holds } w(t) \in \widehat{\mathcal{W}_0} \Big\}. \tag{3.142}$$

3.6. Long Time Behavior

In this section, we analyze the long time behavior of the solution (a, w) to Problem 3.1.2 for the case $\mathbb{T} = \mathbb{S}^1$. We assume that for given data $\alpha \in \mathbb{R}$, $\kappa > 0$ and given initial data $(a_0, w_0) \in L^2(\mathbb{S}^1) \times \mathcal{W}_0$ there exists a solution (a, w) to Problem 3.1.2 in the sense of Definition 3.1.3. We recall the energy balance (3.15) from Section 3.1. Since $f'(w)$ and w always have the same sign and since $\kappa > 0$, the energy balance implies the estimate

$$\mathcal{F}\big(a(t), w(t)\big) \leq \mathcal{F}\big(a(t), w(t)\big) + \kappa \int_0^t \int_{\mathbb{S}^1} f'(w(s))\, w(s)\, dx\, ds = \mathcal{F}(a_0, w_0). \tag{3.143}$$

In particular, due to the definition $\mathcal{F}(a, w) := \int_{\mathbb{S}^1} \frac{1}{2}|a|^2 + f(w)\, dx$ and $f(\cdot) \geq 0$, the energy estimate implies

$$\max_{t \geq 0} \int_{\mathbb{S}^1} |a(x, t)|^2\, dx \leq 2\, \mathcal{F}(a_0, w_0). \tag{3.144}$$

The free energy $\mathcal{F}\big(a(t), w(t)\big)$ is greater or equal to zero for all $t \geq 0$ and the initial free energy $\mathcal{F}(a_0, w_0)$ is finite (due to our choice of the initial data). Moreover, we have $f'(w)\, w \geq 0$. Therefore, we may infer

$$\lim_{t \to \infty} \int_0^t \int_{\mathbb{S}^1} f'(w(x, s))\, w(x, s)\, dx\, ds < \infty. \tag{3.145}$$

Furthermore, the convexity of f and $f(0) = 0$ yield

$$\forall\, w \in [-1, 1] : \; f(0) \geq f(w) + f'(w)(0 - w), \qquad \text{i. e.} \qquad f'(w)\, w \geq f(w) \tag{3.146}$$

Next, we study the long time behavior of the functions w and a separately.

3.6.1. Long Time Behavior of the Inversion w

Our first result in Theorem 3.6.2 concerning the behavior of w for $t \to \infty$ crucially depends on the following Lemma.

Lemma 3.6.1. *Let arbitrary $\delta \geq 0$ and $p \in [2, \infty)$ be given. Then, for all initial data $(a_0, w_0) \in L^2(\mathbb{S}^1) \times \mathcal{W}_0$ with $\|w_0\|^p_{L^p(\mathbb{S}^1)} \leq \delta$, there exists $T_\delta = T_\delta(p) > 0$ such that the solution w to Problem 3.1.2 satisfies*

$$\forall\, t \in [0, T_\delta(p)] : \qquad \|w(t)\|^p_{L^p(\mathbb{S}^1)} \leq 2\delta. \tag{3.147}$$

Proof. For arbitrary $t > 0$ it holds

$$\begin{aligned}
&\int_{\mathbb{S}^1} |w(t)|^p \, dx = \int_{\mathbb{S}^1} \int_0^t \partial_t |w(x,t)|^p \, dt \, dx + \int_{\mathbb{S}^1} |w_0|^p \, dx \\
&= \int_{\mathbb{S}^1} p \int_0^t \Big(\alpha \chi_{\Omega_{\text{act}}} h(w(s)) |a(s)|^2 w(s) |w(s)|^{p-2} - \kappa |w(s)|^p \Big) \, ds + |w_0|^p \, dx \\
&\leq p \, \frac{|\alpha|}{2} \int_0^t \int_{\Omega_{\text{act}}} \big(|a(x,s)|^2 + \kappa \big) \, dx \, ds + \delta \leq p \, |\alpha| \, t \, \big(\mathcal{F}(a_0, w_0) + \kappa \, |\Omega_{\text{act}}| \big) + \delta. \qquad (3.148)
\end{aligned}$$

With $C\big(\mathcal{F}(a_0, w_0), \Omega_{\text{act}}, \alpha, \kappa, p\big) := p \, |\alpha| \, \big(\mathcal{F}(a_0, w_0) + \kappa \, |\Omega_{\text{act}}| \big)$ it is clear that for sufficiently small $T_\delta(p)$, we have that

$$\forall \, t \in [0, T_\delta(p)] : \qquad t \cdot C\big(\mathcal{F}(a_0, w_0), \Omega_{\text{act}}, \alpha, \kappa, p\big) \leq \delta. \tag{3.149}$$

This proves the claim. $\square$

We stress that the above result includes the case $\Omega_{\text{act}} = \mathbb{S}^1$. With Lemma 3.6.1 we may prove the following result concerning the behavior of the inversion w for $t \to \infty$.

Theorem 3.6.2. *For all initial data $(a_0, w_0) \in L^2(\mathbb{S}^1) \times \mathcal{W}_0$ and for all $p \geq 1$ the solution w to Problem 3.1.2 satisfies*

$$\lim_{t \to \infty} \int_{\mathbb{S}^1} |w(x,t)|^p \, dx = 0. \tag{3.150}$$

Proof. Due to Hölder's estimate, it is sufficient to prove the theorem for $p \geq 2$, since for all $t \geq 0$, we have

$$\|w(t)\|_{L^1(\mathbb{S}^1)} \leq \Big(\int_{\mathbb{S}^1} |1|^2 \, dx \Big)^{1/2} \cdot \|w(t)\|_{L^2(\mathbb{S}^1)} = \sqrt{2\pi} \cdot \|w(t)\|_{L^2(\mathbb{S}^1)}. \tag{3.151}$$

To prove the theorem for $p \geq 2$, let some arbitrary but sufficiently small $\delta > 0$ be given and define $\epsilon(\delta) := \frac{\delta T_\delta(p)}{4}$ with $T_\delta(p)$ as in the lemma above. From (3.145) and the estimate $|w|^p \leq f(w) \leq f'(w) \, w$ which, in view of Lemma A.1.20, holds for all $p \in [2, \infty)$, we may infer the existence of some $t_*(\epsilon, \delta, p)$, such that

$$\int_{t_*(\epsilon,\delta,p)}^{\infty} \int_{\mathbb{S}^1} |w(x,s)|^p \, dx \, ds \leq \epsilon(\delta). \tag{3.152}$$

Furthermore, due to Proposition 3.1.4 we have $w \in C^0([0,T]; L^p(\mathbb{S}^1))$ for all $p \in [2, \infty)$. Therefore, for all $p \in [2, \infty)$ there exists some $t_\delta^0 \geq t_*(\epsilon, \delta, p)$ such that $\|w(t_\delta^0)\|_{L^p(\mathbb{S}^1)}^p \leq \delta$. Due to Lemma 3.6.1 we have

$$\forall \, t \in [t_\delta^0, t_\delta^0 + T_\delta(p)] : \qquad \|w(t)\|_{L^p(\mathbb{S}^1)}^p \leq 2\delta. \tag{3.153}$$

In particular, due to the continuity of w in time with values in $L^p(\mathbb{S}^1)$ and due to (3.152) as well as the definition of $\epsilon(\delta)$, there exists some

$$t_\delta^1 \in [t_\delta^0 + \tfrac{3}{4}T_\delta(p), t_\delta^0 + T_\delta(p)] : \qquad \|w(t_\delta^1)\|^p_{L^p(\mathbb{S}^1)} \le \delta. \tag{3.154}$$

By induction, this implies the existence of a sequence $\{t_\delta^k\}_{k\in\mathbb{N}}$ with

$$\forall\, k \in \mathbb{N} : \qquad t_\delta^{k+1} > t_\delta^k, \qquad \left|t_\delta^k - t_\delta^{k+1}\right| \ge \frac{3}{4}\, T_\delta(p), \qquad \left\|w(t_\delta^k)\right\|^p_{L^p(\mathbb{S}^1)} \le \delta. \tag{3.155}$$

Lemma 3.6.1 thus shows that for all $t \ge t_*(\epsilon, \delta, p)$ we have $\|w(t)\|^p_{L^p(\mathbb{S}^1)} \le 2\delta$.
Since $\delta > 0$ can be chosen arbitrarily small, we may infer

$$\|w(t)\|^p_{L^p(\mathbb{S}^1)} \longrightarrow 0 \qquad \text{as } t \to \infty. \tag{3.156}$$

This ends the proof of Theorem 3.6.2, because the argument holds for all $p \in [2, \infty)$. $\square$

3.6.2. Long Time Behavior of the Amplitude a

Next, we analyze the behavior of the amplitude a as $t \to \infty$. We are able to give a statement for the case $\Omega_{\mathrm{act}} \equiv \mathbb{S}^1$, only. Namely, we have the following result.

Proposition 3.6.3. *Let $\Omega_{\mathrm{act}} \equiv \mathbb{S}^1$. Then, for all initial data $(a_0, w_0) \in L^2(\mathbb{S}^1) \times \mathcal{W}_0$ the solution a to Problem 3.1.2 satisfies*

$$\lim_{t\to\infty} \int_{\mathbb{S}^1} |a(x,t)|^2 \, dx = 0. \tag{3.157}$$

The proof of this proposition is based on the following lemma. For notational convenience, we define the set

$$\mathcal{W}_\infty := \Big\{ w \in W^{1,\infty}((0,\infty); L^1(\mathbb{S}^1)) : \ \forall\, t \ge 0 \text{ it holds } w(t) \in \mathcal{W}_0 \Big\}.$$

Lemma 3.6.4. *Let $\alpha \in \mathbb{R}$, $\kappa > 0$ and initial data $(a_0, w_0) \in L^2(\mathbb{S}^1) \times \mathcal{W}_0$ be given. Assume that there exists a global solution $(a, w) \in C^0([0,\infty); L^2(\mathbb{S}^1)) \times \mathcal{W}_\infty$ to Problem 3.1.2. Then, for all $t \ge 0$ the following estimate holds pointwise for a.a. $x \in \mathbb{S}^1$*

$$-\,\alpha \int_0^t w(x + \tau - t, \tau)\, d\tau \le \tfrac{|\alpha|}{\kappa}\big(1 - e^{-\kappa t}\big). \tag{3.158}$$

In particular, we have the following estimate for a.e. $(x,t) \in [0,\infty) \times \mathbb{S}^1$

$$|a(x,t)|^2 \le |a_0(x-t)|^2 \cdot \exp\Big(\tfrac{2\,|\alpha|}{\kappa}\big(1 - e^{-\kappa t}\big)\Big). \tag{3.159}$$

Proof. We assume w.l.o.g. that $\alpha > 0$, define $\widehat{w}_0 := \min\{0, w_0(x)\}$ and consider the following Cauchy problem for fixed $x \in \Omega_{\mathrm{act}}$

$$\partial_t \widehat{w}(x,t) = -\kappa\, \widehat{w}(x,t), \qquad \widehat{w}(0,x) = \widehat{w}_0(x). \tag{3.160}$$

The solution is obviously given by $\widehat{w}(x,t) = \widehat{w}(0,x) \cdot e^{-\kappa t}$. For a.a. $x \in \mathbb{S}^1$, the function w solves the Cauchy problem

$$\begin{aligned} \partial_t w(x,t) &= \alpha\, \chi_{\Omega_{\text{act}}}(x)\, h(w)\, |a(x,t)|^2 - \kappa\, w(x,t) && \text{in } [0,T] \times \mathbb{T} \\ w(0,x) &= w_0(x) && \text{in } \mathbb{T}. \end{aligned}$$

Due to $\alpha h(w)|a|^2 \geq 0$, we have pointwise for a.e. $(x,t) \in [0,T] \times \mathbb{S}^1$ the estimate

$$\partial_t w(x,t) \geq \partial_t \widehat{w}(x,t).$$

Due to $w_0(x) \geq \widehat{w}_0(x)$ for a.a. $x \in \mathbb{S}^1$, this implies

$$\forall_{\text{a.a.}}\, (x,t) \in [0,\infty) \times \mathbb{S}^1 : \qquad w(x,t) \geq \widehat{w}(x,t). \tag{3.161}$$

Furthermore, since for a.e. $x \in \mathbb{S}^1$ it holds $w_0(x) \geq -1$, we may infer the validity of the following estimate

$$\forall_{\text{a.a.}}\, (x,t) \in [0,\infty) \times \mathbb{S}^1 : \qquad w(x,t) \geq \widehat{w}(x,t) = \min\{0, w_0(x)\}\, e^{-\kappa t} \geq -e^{-\kappa t}.$$

Thus, we get the following estimate for all $t \geq 0$ and for a.e. $x \in \mathbb{S}^1$

$$-\alpha \int_0^t w(x+\tau-t,\tau)\, d\tau \leq -\alpha \int_0^t -e^{-\kappa\tau}\, d\tau = \alpha\Big(\frac{1}{\kappa}\big(1-e^{-\kappa t}\big)\Big) \leq \frac{|\alpha|}{\kappa}.$$

From the solution formula (3.51) for the amplitude a, we may infer that the following estimate holds. For a.a. $(x,t) \in [0,\infty) \times \mathbb{S}^1$ we have

$$\begin{aligned} |a(x,t)| &= |a_0(x-t)| \cdot \exp\Big(-\alpha \int_0^t w(x+\tau-t,\tau)\, d\tau\Big) \\ &\leq |a_0(x-t)| \cdot \exp\Big(-\alpha \int_0^t -e^{-\kappa\tau}\, d\tau\Big) \\ &\leq |a_0(x-t)| \cdot \exp\Big(\frac{\alpha}{\kappa}\big(1-e^{-\kappa t}\big)\Big). \end{aligned}$$

Squaring the last estimate yields the assertion. □

Proof of Proposition 3.6.3. We begin with the definition of the time-shifted functions

$$\begin{aligned} \widetilde{a}(x,t) &:= a(x+t,t) = a_0(x) \cdot \exp\Big(-\alpha \int_0^t w(x+\tau,\tau)\, d\tau\Big), \\ \widetilde{w}(x,t) &:= w(x+t,t), \\ \widetilde{\chi}_{\Omega_{\text{act}}}(x,t) &:= \chi_{\Omega_{\text{act}}}(x+t). \end{aligned}$$

Next, we proceed in 3 steps. The last step crucially depends on $\Omega_{\text{act}} \equiv \mathbb{S}^1$.

Step 1: We show that the following quantity $B(s)$ converges to zero as $s \to \infty$

$$B(s) := \int_{\mathbb{S}^1} \chi_{\Omega_{\mathrm{act}}} \big| h(0) - h(w(s)) \big| \cdot |a(s)|^2 \, dx = \int_{\mathbb{S}^1} \widetilde{\chi}_{\Omega_{\mathrm{act}}} \big| h(0) - h(\widetilde{w}(s)) \big| \cdot |\widetilde{a}(s)|^2 \, dx.$$

Due to Lemma 3.6.4 we have pointwise for a.e. $x \in \mathbb{S}^1$ and for all $s \geq 0$ the estimate

$$\begin{aligned} b(x,s) &:= \widetilde{\chi}_{\Omega_{\mathrm{act}}}(x,s) \big| h(0) - h(\widetilde{w}(x,s)) \big| \cdot |\widetilde{a}(x,s)|^2 \\ &\leq 2\, h(0) \cdot |a_0(x)|^2 \exp\left(\tfrac{2\,|\alpha|}{\kappa}\right) =: g(x). \end{aligned} \tag{3.162}$$

Due to our assumptions on the initial datum a_0, we may infer that $g \in L^1(\mathbb{S}^1)$ is a majorant to b, independent of s. In particular, this implies the boundedness of the quantity $B(s)$ for all $s \geq 0$. Due to this boundedness of $B(s)$, it is clear that $B_\infty := \limsup_{s\to\infty} B(s)$ and some sequence $\{s_j\}_{j\in\mathbb{N}} \nearrow \infty$ exist with $\lim_{j\to\infty} B(s_j) = B_\infty$. From the convergence (3.156) we get the existence of some subsequence $\{s_{j_k}\}_{k\in\mathbb{N}}$ that satisfies the following convergence pointwise for a.e. $x \in \mathbb{S}^1$

$$\widetilde{w}(x, s_{j_k}) \longrightarrow 0 \qquad \text{as } k \to \infty. \tag{3.163}$$

This implies the convergence $b(x, s_{j_k}) \longrightarrow 0$ pointwise for a.e. $x \in \mathbb{S}^1$ as $k \to \infty$. Thus, Lebesgue's dominated convergence theorem yields the convergence

$$B(s_{j_k}) \longrightarrow 0 \qquad \text{as } k \to \infty. \tag{3.164}$$

Due to the uniqueness of the limit, this implies $B_\infty = 0$. Moreover, due to

$$0 = B_\infty := \limsup_{s\to\infty} B(s) \geq \lim_{s\to\infty} B(s) \geq 0, \tag{3.165}$$

we may infer $\lim_{s\to\infty} B(s) = 0$.
Step 2: We show that it holds

$$\int_0^t \|a(s)\|^2_{L^2(\Omega_{\mathrm{act}})}\, e^{-\kappa(t-s)} \, ds \longrightarrow 0 \qquad \text{as } t \to \infty. \tag{3.166}$$

The variation of constants formula yields the following equation for the inversion w for all $t \geq 0$ and for a.e. $x \in \mathbb{S}^1$

$$w(x,t) = w_0(x)\, e^{-\kappa\, t} + \alpha \int_0^t \chi_{\Omega_{\mathrm{act}}}(x) h(w(x,s))\, |a(x,s)|^2\, e^{-\kappa(t-s)} \, ds. \tag{3.167}$$

Taking the $L^p(\mathbb{S}^1)$-norm for any $p \in [1,\infty)$ yields with the inverse triangle inequality

$$\|w(t)\|_{L^p(\mathbb{S}^1)} \geq \left| \big\| w_0\, e^{-\kappa\, t} \big\|_{L^p(\mathbb{S}^1)} - \left\| \alpha \int_0^t h(w(s))\, |a(s)|^2\, e^{-\kappa(t-s)} \, ds \right\|_{L^p(\Omega_{\mathrm{act}})} \right|. \tag{3.168}$$

Due to the convergence $\|w(t)\|^p_{L^p(\mathbb{S}^1)} \longrightarrow 0$ as $t \to \infty$ from (3.156) and the obvious convergence $\|w_0\, e^{-\kappa t}\|_{L^p(\mathbb{S}^1)} \longrightarrow 0$ as $t \to \infty$, we may infer the convergence

$$0 = \lim_{t\to\infty} \Big\| \int_0^t h(w(s))\, |a(s)|^2\, e^{-\kappa(t-s)}\, ds \Big\|_{L^p(\Omega_{\mathrm{act}})}. \tag{3.169}$$

Taking into account $h(0) = 1/2$, the inverse triangle inequality yields

$$\begin{aligned} &\Big\| - \int_0^t h(w(s))\, |a(s)|^2\, e^{-\kappa(t-s)}\, ds \Big\|_{L^1(\Omega_{\mathrm{act}})} \\ &\geq \bigg| \Big\| \int_0^t \big|h(0) - h(w(s))\big|\, |a(s)|^2\, e^{-\kappa(t-s)}\, ds \Big\|_{L^1(\Omega_{\mathrm{act}})} - \Big\| \tfrac{1}{2} \int_0^t |a(s)|^2\, e^{-\kappa(t-s)}\, ds \Big\|_{L^1(\Omega_{\mathrm{act}})} \bigg|. \end{aligned}$$

From *Step 1* we know that the first summand in the last line tends to zero in the limit as $t \to \infty$. Together with (3.169), this yields (3.166).
Step 3: We conclude and show the convergence from (3.157). At this point we need $\Omega_{\mathrm{act}} \equiv \mathbb{S}^1$. We recall the energy balance for the amplitude a from (3.13) and consider

$$\begin{aligned} A(t) :=& \Big\| \int_0^t |a(s)|^2\, e^{-\kappa(t-s)}\, ds \Big\|_{L^1(\mathbb{S}^1)} = \int_0^t \|a(s)\|^2_{L^2(\mathbb{S}^1)}\, e^{-\kappa(t-s)}\, ds \\ =& \frac{1}{\kappa} \Big[\|a(s)\|^2_{L^2(\mathbb{S}^1)}\, e^{-\kappa(t-s)} \Big]_{s=0}^{s=t} - \frac{2\,\alpha}{\kappa} \int_0^t \int_{\mathbb{S}^1} w(s)\, |a(s)|^2\, e^{-\kappa(t-s)}\, ds\, dx \\ \geq& \frac{1}{\kappa} \Big(\|a(t)\|^2_{L^2(\mathbb{S}^1)} - e^{-\kappa t}\, \|a_0\|^2_{L^2(\mathbb{S}^1)} \Big) - \frac{2\,|\alpha|}{\kappa} A(t). \end{aligned} \tag{3.170}$$

In the case $\Omega_{\mathrm{act}} \equiv \mathbb{S}^1$, the convergence from (3.166) is equivalent to $A(t) \longrightarrow 0$ as $t \to \infty$. Therefore, the above estimate implies

$$\|a(t)\|^2_{L^2(\mathbb{S}^1)} \leq (\kappa + 2|\alpha|)\, A(t) + e^{-\kappa t} \|a_0\|^2_{L^2(\mathbb{S}^1)} \longrightarrow 0 \qquad \text{as } t \to \infty. \tag{3.171}$$

This finishes the proof of Proposition 3.6.3. $\square$

4. Analysis of a Dissipative Maxwell-Bloch Type Model

In Section 2.5 we derived the following system of differential equations that models the interaction of some *active material* described by the Bloch vector $\boldsymbol{a}$ and *light* described by the electromagnetic field $\big(\mathbf{E}, \mathbf{H}\big)$

$$\partial_t \mathbf{E} = \epsilon_0^{-1} \operatorname{curl} \mathbf{H} - \epsilon_0^{-1} \mathsf{G} \partial_t \boldsymbol{a} \tag{4.1a}$$

$$\partial_t \mathbf{H} = -\mu_0^{-1} \operatorname{curl} \mathbf{E} \tag{4.1b}$$

$$\begin{aligned} \partial_t \boldsymbol{a} = {} & \frac{1}{\hbar}\big(\boldsymbol{a} \times (2\mathbf{h} - \mathsf{G}^* \mathbf{E})\big) \\ & + \frac{2}{\vartheta_*} \sum_{m=1}^{M} \mathbf{q}_m \times \mathsf{C}_{\boldsymbol{a}} \Big(\mathbf{q}_m \times \Big(\big(2\mathbf{h} - \mathsf{G}^* \mathbf{E}\big) + 2\, \vartheta_* k_B \frac{\boldsymbol{a}}{\lambda(|\boldsymbol{a}|)} \Big) \Big). \end{aligned} \tag{4.1c}$$

In this system the canonical correlation operator $\mathsf{C}_{\boldsymbol{a}}$ is defined by

$$\mathsf{C}_{\boldsymbol{a}} := \lambda(|\boldsymbol{a}|) \mathsf{Id}_{3\times 3} + \mu(|\boldsymbol{a}|)\, \boldsymbol{a} \otimes \boldsymbol{a} \tag{4.2}$$

involving the non-linear functions $\lambda : \mathbb{R} \longrightarrow [0, 1/2]$ and $\mu : \mathbb{R} \longrightarrow [0, 2]$ defined by

$$\lambda(|\boldsymbol{a}|) := \begin{cases} \frac{2|\boldsymbol{a}|}{\log(1/2 + |\boldsymbol{a}|) - \log(1/2 - |\boldsymbol{a}|)}, & |\boldsymbol{a}| < 1/2 \\ 0, & |\boldsymbol{a}| \geq 1/2 \end{cases}, \qquad \mu(|\boldsymbol{a}|) := \frac{1 - 2\lambda(|\boldsymbol{a}|)}{2|\boldsymbol{a}|^2}. \tag{4.3}$$

In the following, we will consider the Cauchy problem for (4.1) with given initial data $\big(\mathbf{E}_0, \mathbf{H}_0, \boldsymbol{a}_0\big)$. We will not analyze this Cauchy problem in the state space introduced in Section 2.4. Instead, we will give an existence theorem for a regularized version of the problem in the space $L^2(\mathbb{R}^3, \mathbb{R}^3_{\mathbf{E}}) \times L^2(\mathbb{R}^3, \mathbb{R}^3_{\mathbf{H}}) \times (L^2 \cap L^\infty)(\Omega_{\text{act}}, \mathbb{R}^3_{\boldsymbol{a}})$. Moreover, we request that, for the Maxwell part of the system, charge and current conservation hold. This means we have to require

$$\operatorname{div}\big(\mathbf{H}_0\big) = 0, \qquad \operatorname{div}\big(\epsilon_0 \mathbf{E}_0 + \mathsf{G}\boldsymbol{a}_0\big) = 0 \tag{4.4}$$

for the initial data. At least formally, charge and current conservation hold for all times, if they are satisfied initially. To see this, consider smooth functions $\big(\mathbf{E}, \mathbf{H}, \boldsymbol{a}\big)$ and apply div to (2.129a) and (2.129b). Due to $\operatorname{div}\operatorname{curl} \equiv 0$, this yields the assertion.

This chapter is organized as follows. In the first section we essentially introduce the notion of solutions we want to consider and give a Lipschitz continuous approximation of the non-linear non-Lipschitzean functions λ and μ. Our results are stated in Section 4.2. The third section is devoted to the proof of the theorems from Section 4.2. The approach of the proof is quite similar to the one from Section 3.4, but far more involved. In contrast to the proof from Section 3.4 we can only attack a certain part of some resulting term with arguments from compensated compactness, but have to argue otherwise for the rest. Besides the usage of a result from compensated compactness, the main highlight of the proof here is the application of a Rellich-type lemma (see Section 4.3.4 and Theorem A.3.14).

Furthermore, we establish the following notation for this chapter. The Bloch vector $\boldsymbol{a}$ is always $\mathbb{R}^3_{\boldsymbol{a}}$-valued, the electric field $\mathbf{E}$ is always $\mathbb{R}^3_{\mathbf{E}}$-valued and the magnetic field $\mathbf{H}$ is always $\mathbb{R}^3_{\mathbf{H}}$-valued. For brevity, we will usually denote the function spaces for $\mathbf{E}$, $\mathbf{H}$ and $\boldsymbol{a}$, i.e. $L^2(\mathbb{R}^3, \mathbb{R}^3_{\mathbf{E}})$, $L^2(\mathbb{R}^3, \mathbb{R}^3_{\mathbf{H}})$ and $(L^2 \cap L^\infty)(\Omega_{\text{act}}, \mathbb{R}^3_{\boldsymbol{a}})$, with $L^2(\mathbb{R}^3)$ and $(L^2 \cap L^\infty)(\Omega_{\text{act}})$, respectively. Moreover, in the whole space case we will usually denote the spaces $H^s(\mathbb{R}^3)$, $s \in \mathbb{R}$, with H^s and the corresponding norms with $\|\cdot\|_{H^s}$. In particular, we will usually denote $L^2(\mathbb{R}^3)$ with L^2 and the corresponding norm with $\|\cdot\|_{L^2}$. In contrast, spaces such as $L^2(\Omega)$ for $\Omega \neq \mathbb{R}^3$ are denoted with $L^2(\Omega)$ at all times. For all other functions appearing, we will always state their image spaces.

4.1. Mathematical Formulation of the Problem

In this section, we give a mathematical formulation of the Cauchy problem for (4.1). In Section 4.1.1 we fix the data of the problem and state in which sense the condition (4.4) is to be satisfied. In Section 4.1.2 we regularize the non-linearity $\mathsf{C}_{\boldsymbol{a}}$ and define a general class of functions we admit as right hand sides in (4.1c). Finally, in Section 4.1.3 we introduce our notion of solutions to the (regularized) Cauchy problem for (4.1) and give some a-priori statements on possible solutions.

4.1.1. Assumption on the Data

We assume that the active material described by $\boldsymbol{a}$ occupies only a certain finite part Ω_{act} of the whole space $\mathbb{R}^3$, and the fields $\mathbf{E}$ and $\mathbf{H}$ are defined in the whole space $\mathbb{R}^3$. Thus, we fix the domain

$$\Omega_{\text{act}} \subsetneq \mathbb{R}^3, \qquad \text{open and bounded.} \tag{4.5}$$

In order to get a proper definition of the dipole moment operator G^* and its adjoint G in this context[1], we introduce the $\mathcal{L}(\mathbb{R}^3_{\mathbf{E}}, \mathbb{R}^3_{\boldsymbol{a}})$-valued function

$$\gamma^* \in L^\infty(\Omega_{\text{act}}; \mathcal{L}(\mathbb{R}^3_{\mathbf{E}}, \mathbb{R}^3_{\boldsymbol{a}})). \tag{4.6}$$

[1] Since L^2 is reflexive, we can interpret G to be the adjoint of G^*, in contrast to the definitions in (2.97) and (2.99).

Then, we define the dipole moment operator G^* by

$$\mathsf{G}^* : \begin{cases} L^p(\mathbb{R}^3;\mathbb{R}^3_{\mathbf{E}}) \longrightarrow L^p(\Omega_{\text{act}};\mathbb{R}^3_{\boldsymbol{a}}) \\ \mathbf{E} \longmapsto \mathsf{G}^*\mathbf{E} : \begin{cases} \Omega_{\text{act}} \longrightarrow \mathbb{R}^3_{\boldsymbol{a}} \\ x \longmapsto \gamma^*(x)\,\mathbf{E}(x) \end{cases} \end{cases} \qquad \text{for all } p \in [1,\infty]. \tag{4.7}$$

This implies that the adjoint of the dipole moment operator denoted by G is given by

$$\mathsf{G} : \begin{cases} L^p(\Omega_{\text{act}};\mathbb{R}^3_{\boldsymbol{a}}) \longrightarrow L^p(\mathbb{R}^3;\mathbb{R}^3_{\mathbf{E}}) \\ \boldsymbol{a} \longmapsto \mathsf{G}\boldsymbol{a} : \begin{cases} \mathbb{R}^3 \longrightarrow \mathbb{R}^3_{\mathbf{E}} \\ x \longmapsto \begin{cases} \gamma(x)\,\boldsymbol{a}(x), & x \in \Omega_{\text{act}} \\ 0 & x \notin \Omega_{\text{act}} \end{cases} \end{cases} \end{cases} \qquad \text{for all } p \in [1,\infty] \tag{4.8}$$

where the $\mathcal{L}(\mathbb{R}^3_{\boldsymbol{a}},\mathbb{R}^3_{\mathbf{E}})$-valued function γ is the transposed of the function γ^*. Clearly, it holds

$$\|\gamma^*\|_{L^\infty(\Omega_{\text{act}};\mathcal{L}(\mathbb{R}^3_{\mathbf{E}},\mathbb{R}^3_{\boldsymbol{a}}))} = \|\gamma\|_{L^\infty(\Omega_{\text{act}};\mathcal{L}(\mathbb{R}^3_{\boldsymbol{a}},\mathbb{R}^3_{\mathbf{E}}))} =: C_{\mathsf{G}}. \tag{4.9}$$

Therefore, it is straightforward to see that for all $p \in [1,\infty]$ we have

$$\left\|\mathsf{G}^*\right\|_{\mathcal{L}(L^p(\mathbb{R}^3),L^p(\Omega_{\text{act}}))} \leq C_{\mathsf{G}}, \qquad \left\|\mathsf{G}\right\|_{\mathcal{L}(L^p(\Omega_{\text{act}}),L^p(\mathbb{R}^3))} \leq C_{\mathsf{G}}. \tag{4.10}$$

For the $\mathbb{R}^3_{\boldsymbol{a}}$-valued functions $\mathbf{h}$, $\mathbf{q}_m$, $m \in \{1,\dots,M\}$ appearing in (4.1c) we assume essential boundedness. Therefore let

$$\mathbf{h} \in L^\infty(\Omega_{\text{act}};\mathbb{R}^3_{\boldsymbol{a}}), \qquad \mathbf{q}_m \in L^\infty(\Omega_{\text{act}};\mathbb{R}^3_{\boldsymbol{a}}). \tag{4.11}$$

In the most general case one would assume that the electric permittivity ϵ_0 and the magnetic permeability μ_0 are essentially bounded $\mathcal{L}(\mathbb{R}^{3\times3},\mathbb{R}^{3\times3})$-valued tensors, i.e.[2]

$$\epsilon_0 \in L^\infty(\mathbb{R}^3;\mathcal{L}(\mathbb{R}^{3\times3}_{\mathbf{E}},\mathbb{R}^{3\times3}_{\mathbf{E}})), \qquad \mu_0 \in L^\infty(\mathbb{R}^3;\mathcal{L}(\mathbb{R}^{3\times3}_{\mathbf{H}},\mathbb{R}^{3\times3}_{\mathbf{H}})).$$

We restrict our analysis to the case ϵ_0, $\mu_0 > 0$. Thus, we fix the two constants

$$\epsilon_0 > 0, \qquad \mu_0 > 0 \tag{4.12}$$

and recall that in [Joc02b] and [DuS12] related systems with variable ϵ_0, μ_0 are studied. For future reference, we fix the following condition.

Condition 4.1.1. *Let a final time $T > 0$ be given and assume* (4.5), (4.6) *as well as* (4.12). *Moreover, let the dipole moment operator G^* and its adjoint G be defined according to* (4.7) *and* (4.8).

[2]In this general case, the quantities would be denoted with ϵ_r and μ_r instead of ϵ_0 and μ_0.

In the considered case, where $\boldsymbol{a}$ is defined in $\Omega_{\text{act}} \subsetneq \mathbb{R}^3$, only, and $\mathbf{E}$, $\mathbf{H}$ are defined in the whole space $\mathbb{R}^3$, the free energy functional is given by

$$\widehat{\mathcal{F}}_{\vartheta_*}\big(\mathbf{E},\mathbf{H},\boldsymbol{a}\big) = \int_{\mathbb{R}^3} \frac{1}{2}\epsilon_0\big|\mathbf{E}\big|^2 + \frac{1}{2}\mu_0\big|\mathbf{H}\big|^2\,dx + 2\int_{\Omega_{\text{act}}} \big(\frac{1}{2}\mathrm{h}_0 + \boldsymbol{a}\cdot\mathbf{h}\big) + \vartheta_* k_B \sigma(|\boldsymbol{a}|)\,dx. \tag{4.13}$$

We choose the initial data $\big(\mathbf{E}_0,\mathbf{H}_0,\boldsymbol{a}_0\big)$ in such a way that for the above free energy functional $\widehat{\mathcal{F}}_{\vartheta_*}$ we have

$$\widehat{\mathcal{F}}_{\vartheta_*}\big(\mathbf{E}_0,\mathbf{H}_0,\boldsymbol{a}_0\big) < \infty. \tag{4.14}$$

Furthermore, we introduce the Helmholtz decomposition for the case $d = 3$

$$L^2(\mathbb{R}^3;\mathbb{R}^3) = L^2_{\|}(\mathbb{R}^3;\mathbb{R}^3) \oplus L^2_{\perp}(\mathbb{R}^3;\mathbb{R}^3) \tag{4.15}$$

and the corresponding projectors $\mathsf{P}_{\|}$ and $\mathsf{P}_{\perp}$ that satisfy

$$\mathsf{P}_{\|} L^2(\mathbb{R}^3;\mathbb{R}^3) = L^2_{\|}(\mathbb{R}^3;\mathbb{R}^3), \qquad \mathsf{P}_{\perp} L^2(\mathbb{R}^3;\mathbb{R}^3) = L^2_{\perp}(\mathbb{R}^3;\mathbb{R}^3). \tag{4.16}$$

In the Helmholtz decomposition $L^2(\mathbb{R}^3;\mathbb{R}^3)$ is decomposed into the irrotational (curl-free) part $L^2_{\|}(\mathbb{R}^3;\mathbb{R}^3)$ and the solenoidal (div-free) part $L^2_{\perp}(\mathbb{R}^3;\mathbb{R}^3)$. For further details on the Helmholtz decomposition we refer to Chapter A.4.

Having introduced the Helmholtz decomposition, we request that charge and current conservation, i.e. the conditions from (4.4), have to be satisfied in the sense that it holds

$$\mathsf{P}_{\|}\mathbf{H}_0 = 0, \qquad \mathsf{P}_{\|}(\epsilon_0\mathbf{E}_0 + \mathsf{G}\boldsymbol{a}_0) = 0. \tag{4.17}$$

Next, we introduce the sets

$$\begin{aligned} \mathbf{L}_{\text{div}} := \Big\{ \big(\mathbf{E}_0,\mathbf{H}_0,\boldsymbol{a}_0\big) \in L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times (L^2 \cap L^\infty)(\Omega_{\text{act}}) :& \\ \mathsf{P}_{\|}(\mathbf{H}_0) = 0, \quad \mathsf{P}_{\|}(\epsilon_0\mathbf{E}_0 + \mathsf{G}\boldsymbol{a}_0) = 0 \Big\},& \qquad (4.18) \\ \mathcal{A}_0 := \Big\{ \boldsymbol{a}_0 \in L^\infty(\Omega_{\text{act}}) : \|\boldsymbol{a}_0\|_{L^\infty(\Omega_{\text{act}})} \leq \tfrac{1}{2} \Big\}.& \qquad (4.19) \end{aligned}$$

Then, our set of admissible initial data is given by the following set

$$\mathcal{IC} := \Big\{ \big(\mathbf{E}_0,\mathbf{H}_0,\boldsymbol{a}_0\big) \in \mathbf{L}_{\text{div}} : \ \boldsymbol{a}_0 \in \mathcal{A}_0 \Big\}. \tag{4.20}$$

In order to allow for a shorter notation and for future reference, we introduce the function $\mathrm{F} : \Omega_{\text{act}} \times \mathbb{R}^3_{\boldsymbol{a}} \times \mathbb{R}^3_{\boldsymbol{a}} \longrightarrow \mathbb{R}^3_{\boldsymbol{a}}$ by

$$\begin{aligned} \mathrm{F}(x,\boldsymbol{a},\mathbf{g_E}) := &\frac{1}{\hbar}\boldsymbol{a} \times \big(2\mathbf{h} - \mathbf{g_E}\big) \\ &+ \frac{2}{\vartheta_*}\sum_{m=1}^{M} \mathbf{q}_m \times \mathsf{C}_{\boldsymbol{a}}\Big(\mathbf{q}_m \times \Big(\big(2\mathbf{h} - \mathbf{g_E}\big) + 2\,\vartheta_* k_B \frac{\boldsymbol{a}}{\lambda(|\boldsymbol{a}|)}\Big)\Big). \end{aligned} \tag{4.21}$$

The x-dependence of the above function is due to the x-dependence of $\mathbf{h}$ and $\mathbf{q}_m$. We make the following observation for this function F.

Lemma 4.1.2. *Let* $\mathbf{h}$, $\mathbf{q}_m \in L^\infty(\Omega_{\text{act}};\mathbb{R}^3_{\boldsymbol{a}})$ *be given.*

(i) If $|\boldsymbol{a}| \geq 1/2$, *for a.a.* $x \in \Omega_{\text{act}}$ *and for all* $\mathbf{g_E} \in \mathbb{R}^3_{\boldsymbol{a}}$ *we have*

$$\mathrm{F}(x, \boldsymbol{a}, \mathbf{g_E}) \cdot \boldsymbol{a} \leq 0. \tag{4.22}$$

(ii) There exists a constant $C_g = C_g(\mathbf{q}_m, \mathbf{h})$ *such that*

$$\forall\, \boldsymbol{a}, \mathbf{g_E} \in \mathbb{R}^3_{\boldsymbol{a}}, \quad \forall_{\text{a.a.}}\, x \in \Omega_{\text{act}}: \qquad \big|\mathrm{F}(x, \boldsymbol{a}, \mathbf{g_E})\big| \leq C_g\big(1 + |\boldsymbol{a}|\big)\big(1 + |\mathbf{g_E}|\big). \tag{4.23}$$

Proof. We recall the miracle relation $\mathsf{C}_{\boldsymbol{a}}\big(\mathbf{q}_m \times \frac{\boldsymbol{a}}{\lambda(|\boldsymbol{a}|)}\big) = \big(\mathbf{q}_m \times \boldsymbol{a}\big)$ from (A.62) which implies

$$\begin{aligned}\mathrm{F}(x,\boldsymbol{a}, \mathbf{g_E}) &:= \frac{1}{\hbar}\big(\boldsymbol{a} \times (2\mathbf{h} - \mathbf{g_E})\big)\\ &\quad + 2\sum_{m=1}^{M} \Big(k_B\big(\mathbf{q}_m \times (\mathbf{q}_m \times 2\boldsymbol{a})\big) + \frac{1}{\vartheta_*}\big(\mathbf{q}_m \times \mathsf{C}_{\boldsymbol{a}}\big(\mathbf{q}_m \times (2\mathbf{h} - \mathbf{g_E})\big)\big)\Big).\end{aligned} \tag{4.24}$$

(i) Multiplying $\mathrm{F}(x, \boldsymbol{a}, \mathbf{g_E})$ with an arbitrary $\boldsymbol{a}$ yields for a.a. $x \in \Omega_{\text{act}}$

$$\mathrm{F}(x, \boldsymbol{a}, \mathbf{g_E}) \cdot \boldsymbol{a} = \sum_{m=1}^{M} k_B\big(\mathbf{q}_m \times 2\boldsymbol{a}\big) \cdot \big(2\boldsymbol{a} \times \mathbf{q}_m\big) + \frac{1}{\vartheta_*}\mathsf{C}_{\boldsymbol{a}}\big(\mathbf{q}_m \times (2\mathbf{h} - \mathbf{g_E})\big) \cdot \big(2\boldsymbol{a} \times \mathbf{q}_m\big).$$

If $|\boldsymbol{a}| \geq 1/2$ we have $\lambda(|\boldsymbol{a}|) = 0$. Thus, from $\mathsf{C}_{\boldsymbol{a}} = \lambda(|\boldsymbol{a}|)\,\mathsf{Id}_{3\times 3} + \mu(|\boldsymbol{a}|)\,\boldsymbol{a} \otimes \boldsymbol{a}$ we get

$$= \sum_{m=1}^{M} -k_B\big|2\boldsymbol{a} \times \mathbf{q}_m\big|^2 + \frac{\mu(|\boldsymbol{a}|)}{\vartheta_*}\boldsymbol{a} \otimes \boldsymbol{a}\big(\mathbf{q}_m \times (2\mathbf{h} - \mathbf{g_E})\big) \cdot \big(2\boldsymbol{a} \times \mathbf{q}_m\big)$$

The symmetry of $\boldsymbol{a} \otimes \boldsymbol{a}$ and the relation $(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{c} = (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$ yield

$$\begin{aligned}&= \sum_{m=1}^{M} -k_B\big|2\boldsymbol{a} \times \mathbf{q}_m\big|^2 + \frac{\mu(|\boldsymbol{a}|)}{\vartheta_*}\big(\mathbf{q}_m \times (2\mathbf{h} - \mathbf{g_E})\big) \cdot \big(\boldsymbol{a} \cdot \big(2\boldsymbol{a} \times \mathbf{q}_m\big)\big)\boldsymbol{a}\\ &= \sum_{m=1}^{M} -k_B\big|2\boldsymbol{a} \times \mathbf{q}_m\big|^2 \leq 0.\end{aligned} \tag{4.25}$$

(ii) Since $\mathsf{C}_{\boldsymbol{a}}$ is uniformly (in $\boldsymbol{a}$) bounded, this estimate is obvious. $\square$

In the next subsection we introduce a regularization of the non-linear non-Lipschitzean function F in terms of a (non-linear) Lipschitz continuous approximation.

4.1.2. Regularization and Generalization of the Nonlinearity

We recall that the canonical correlation operator $\mathsf{C}_{\boldsymbol{a}}$ in Bloch coordinates has the representation

$$\mathsf{C}_{\boldsymbol{a}} := \lambda(|\boldsymbol{a}|)\,\mathsf{Id}_{3\times 3} + \mu(|\boldsymbol{a}|)\,\boldsymbol{a}\otimes\boldsymbol{a} \tag{4.26}$$

with the functions λ and μ defined by

$$\lambda(|\boldsymbol{a}|) := \begin{cases} \frac{2|\boldsymbol{a}|}{\log(1/2+|\boldsymbol{a}|)-\log(1/2-|\boldsymbol{a}|)}, & |\boldsymbol{a}| < 1/2 \\ 0, & |\boldsymbol{a}| \geq 1/2 \end{cases}, \qquad \mu(|\boldsymbol{a}|) := \frac{1-2\lambda(|\boldsymbol{a}|)}{2|\boldsymbol{a}|^2}. \tag{4.27}$$

The functions λ, μ are continuous in $[0,1/2]$ and Lipschitz continuous in $[0,1/2-\epsilon]$ for all $\epsilon \ll 1$ and we recall the asymptotics

$$\lim_{r\to 0}\lambda(r) = \frac{1}{2}, \qquad \lim_{r\to 1/2}\lambda(r) = 0 \tag{4.28}$$

$$\lim_{r\to 0}\mu(r) = \frac{2}{3}, \qquad \lim_{r\to 1/2}\mu(r) = 2. \tag{4.29}$$

The circumstance that the functions λ and μ are not Lipschitz continuous on $[0,1/2]$ turned out to make the analysis of an existence proof for the Cauchy problem to (4.1) too difficult to be treated. Therefore, we replace λ and μ with Lipschitz continuous approximations λ_δ and μ_δ. We take a function λ_δ that satisfies the following hypothesis.

Hypothesis 4.1.3. *For a given $0 < \delta \ll 1/2$, let λ_δ be a Lipschitz continuous function on $\mathbb{R}_+$ with the following properties.*

(i) For all $r \in [0,1/2)$ it holds $0 < \lambda_\delta(r) \leq \lambda(r)$.

(ii) $\lambda_\delta(r) = \begin{cases} \lambda(r) & \textit{for } r \in [0,1/2-\delta] \\ 0 & \textit{for } r \geq 1/2. \end{cases}$

(iii) $\mathrm{Lip}(\lambda_\delta) \leq 2\frac{\lambda(1/2-\delta)}{\delta}$. .

The quantity $0 < \delta \ll 1/2$ from the above hypothesis will be referred to as the *regularization parameter.*

Remark 4.1.4. *An example for a function λ_δ that satisfies the above hypothesis can be constructed in the same manner as the approximation h_δ in Section 3.1.2. However, we do explicitly not want to restrict our analysis to this case, because in view of Conjecture 4.4.1 we also want to admit smooth approximations. In particular, we choose a clear bound on the Lipschitz constant* $\mathrm{Lip}(\lambda_\delta)$ *in such a way that it is actually possible to construct smooth approximations. We emphasize that this would not have been possible with the bound* $\mathrm{Lip}(\lambda_\delta) \leq \frac{\lambda(1/2-\delta)}{\delta}$.

Next, we define the function $\mu_\delta(r) := \frac{1-2\lambda_\delta(r)}{2r^2}$ and the regularized canonical correlation operator $\mathsf{C}^\delta_{\boldsymbol{a}}$ via

$$\mathsf{C}^\delta_{\boldsymbol{a}} := \lambda_\delta(|\boldsymbol{a}|)\,\mathsf{Id}_{3\times 3} + \mu_\delta(|\boldsymbol{a}|)\,\boldsymbol{a}\otimes\boldsymbol{a}. \tag{4.30}$$

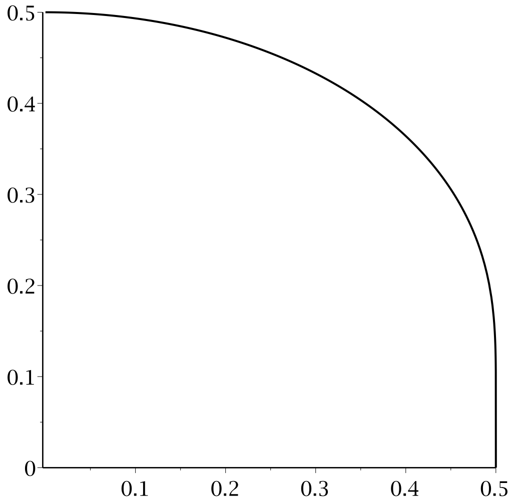

Figure 4.1.: plot of the function λ

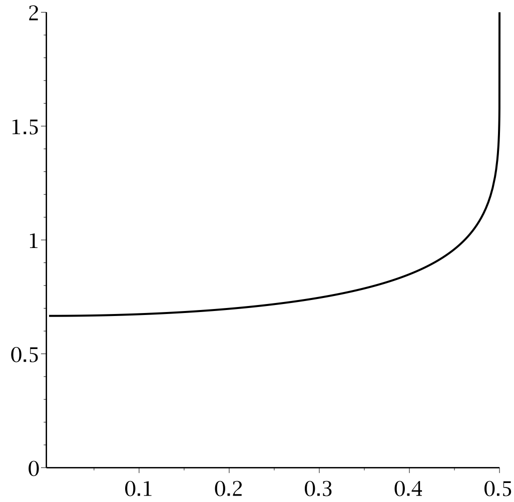

Figure 4.2.: plot of the function μ

The system that results from replacing $\mathsf{C}_{\boldsymbol{a}}$ with $\mathsf{C}^{\delta}_{\boldsymbol{a}}$ can also be written in the form of a damped Hamiltonian system in the sense of Definition 2.1.1. To this end we define the function

$$\sigma_\delta(|\boldsymbol{a}|) := \frac{1}{4}\log(1/4) + \int_0^{|\boldsymbol{a}|} \frac{r}{\lambda_\delta(r)}\,dr, \qquad |\boldsymbol{a}| < 1/2. \tag{4.31}$$

We continuously extend σ_δ to $|\boldsymbol{a}| = 1/2$ and for $|\boldsymbol{a}| > 1/2$ we set $\sigma_\delta(|\boldsymbol{a}|) := \infty$. Then, we define the regularized free energy functional

$$\widehat{\mathcal{F}}^{\delta}_{\vartheta_*}\big(\mathbf{E},\mathbf{H},\boldsymbol{a}\big) := \int_{\mathbb{R}^3} \frac{\epsilon_0}{2}|\mathbf{E}|^2 + \frac{\mu_0}{2}|\mathbf{H}|^2\,dx + 2\int_{\Omega_{\mathrm{act}}} \mathbf{h}\cdot\boldsymbol{a} + k_B\vartheta_*\sigma_\delta(|\boldsymbol{a}|)\,dx. \tag{4.32}$$

The derivative $D\widehat{\mathcal{F}}^{\delta}_{\vartheta_*}$ of the functional $\widehat{\mathcal{F}}^{\delta}_{\vartheta_*}$ is given by

$$D\widehat{\mathcal{F}}^{\delta}_{\vartheta_*}\big(\mathbf{E},\mathbf{H},\boldsymbol{a}\big) = \Big(\epsilon_0\mathbf{E},\ \mu_0\mathbf{H},\ \big(2\mathbf{h} + 2\vartheta_* k_B\frac{\boldsymbol{a}}{\lambda_\delta(|\boldsymbol{a}|)}\big)\Big) \tag{4.33}$$

since obviously, the function σ_δ satisfies $\frac{d}{d\boldsymbol{a}}\sigma_\delta(|\boldsymbol{a}|) = \frac{\boldsymbol{a}}{\lambda_\sigma(|\boldsymbol{a}|)}$. Setting

$$\widehat{\mathcal{K}}_\delta(\boldsymbol{a})[\xi] := -2\sum_{m=1}^{M} \mathbf{q}_m \times \mathsf{C}^{\delta}_{\boldsymbol{a}}\big(\mathbf{q}_m\times\xi\big), \tag{4.34}$$

and defining the regularized Onsager structure $\widehat{\mathbb{K}}_\delta$ by replacing $\widehat{\mathcal{K}}$ with $\widehat{\mathcal{K}}_\delta$ in the definition of $\mathbb{K}\big(\mathbf{E},\mathbf{H},\boldsymbol{a}\big)$ from (2.127), but keeping the original Poisson structure $\widehat{\mathbb{J}}$ from (2.126) we get the damped Hamiltonian system

$$\big(\partial_t\mathbf{E},\partial_t\mathbf{H},\partial_t\boldsymbol{a}\big) = \Big(\widehat{\mathbb{J}}\big(\mathbf{E},\mathbf{H},\boldsymbol{a}\big) - \frac{1}{\vartheta_*}\widehat{\mathbb{K}}_\delta\big(\mathbf{E},\mathbf{H},\boldsymbol{a}\big)\Big)D\widehat{\mathcal{F}}^{\delta}_{\vartheta_*}\big(\mathbf{E},\mathbf{H},\boldsymbol{a}\big). \tag{4.35}$$

Clearly, for every $\boldsymbol{a} \in \mathbb{R}^3_{\boldsymbol{a}}$, the matrix $\mathsf{C}^\delta_{\boldsymbol{a}}$ is symmetric. Moreover, performing the same calculations as in Lemma A.2.3 shows that for every $\boldsymbol{a} \in \mathbb{R}^3_{\boldsymbol{a}}$, the matrix $\mathsf{C}^\delta_{\boldsymbol{a}}$ is positive semi-definite. Therefore, the regularized Onsager operator $\widehat{\mathbb{K}}_\delta$ satisfies the conditions (2.1) and (2.2b) from Definition 2.1.1. For brevity we define the Lipschitz continuous regularization $\mathrm{F}_\delta : \Omega_{\mathrm{act}} \times \mathbb{R}^3_{\boldsymbol{a}} \times \mathbb{R}^3_{\boldsymbol{a}} \longrightarrow \mathbb{R}^3_{\boldsymbol{a}}$ of the function F from (4.21) by

$$\begin{aligned}\mathrm{F}_\delta(x, \boldsymbol{a}, \mathsf{G}^*\mathbf{E}) :=\ & \frac{1}{\hbar}\boldsymbol{a} \times \big(2\mathbf{h} - \mathsf{G}^*\mathbf{E}\big) \\ & + \frac{2}{\vartheta_*}\sum_{m=1}^{M} \mathbf{q}_m \times \mathsf{C}^\delta_{\boldsymbol{a}}\Big(\mathbf{q}_m \times \Big(\big(2\mathbf{h} - \mathsf{G}^*\mathbf{E}\big) + 2\,\vartheta_* k_B \frac{\boldsymbol{a}}{\lambda_\delta(|\boldsymbol{a}|)}\Big)\Big). \end{aligned} \tag{4.36}$$

For the regularized canonical correlation operator $\mathsf{C}^\delta_{\boldsymbol{a}}$ from (4.30), we have an analogue to the miracle relation (A.62). It holds

$$\forall\, \boldsymbol{a} \in \mathbb{R}^3_{\boldsymbol{a}} : \qquad \mathsf{C}^\delta_{\boldsymbol{a}}\big(\mathbf{q}_m \times \tfrac{\boldsymbol{a}}{\lambda_\delta(|\boldsymbol{a}|)}\big) = \big(\mathbf{q}_m \times \boldsymbol{a}\big). \tag{4.37}$$

Therefore, the function F_δ can be written as

$$\begin{aligned}\mathrm{F}_\delta(x, \boldsymbol{a}, \mathsf{G}^*\mathbf{E}) =\ & \frac{1}{\hbar}\big(\boldsymbol{a} \times (2\mathbf{h} - \mathsf{G}^*\mathbf{E})\big) \\ & + 2\sum_{m=1}^{M}\Big(k_B\big(\mathbf{q}_m \times (\mathbf{q}_m \times 2\boldsymbol{a})\big) + \frac{1}{\vartheta_*}\big(\mathbf{q}_m \times \mathsf{C}^\delta_{\boldsymbol{a}}\big(\mathbf{q}_m \times (2\mathbf{h} - \mathsf{G}^*\mathbf{E})\big)\big)\Big). \end{aligned} \tag{4.38}$$

The algebraic properties of the function F_δ become more obvious in this representation. In particular, the results from Lemma 4.1.2 literally hold for the function F_δ.

Next, we collect all relevant properties of the function F_δ and generalize the above situation for an existence proof. Namely, instead of considering the concrete function F_δ from above, we consider a general function f that satisfies the following hypothesis.

Hypothesis 4.1.5. *Let $\mathcal{F}_{\mathrm{hyp}}$ denote the set of functions $f : \Omega_{\mathrm{act}} \times \mathbb{R}^3_{\boldsymbol{a}} \times \mathbb{R}^3_{\boldsymbol{a}} \longrightarrow \mathbb{R}^3_{\boldsymbol{a}}$ that satisfy the following properties.*

(i) There exist functions $f_0 : \Omega_{\mathrm{act}} \times \mathbb{R}^3_{\boldsymbol{a}} \longrightarrow \mathbb{R}^3_{\boldsymbol{a}}$ and $f_1 : \Omega_{\mathrm{act}} \times \mathbb{R}^3_{\boldsymbol{a}} \longrightarrow \mathcal{L}(\mathbb{R}^3_{\boldsymbol{a}}, \mathbb{R}^3_{\boldsymbol{a}})$ with $f(x, \boldsymbol{a}, \mathbf{g}_{\mathbf{E}}) = f_0(x, \boldsymbol{a}) + f_1(x, \boldsymbol{a})\mathbf{g}_{\mathbf{E}}$.

(ii) The functions f_0, f_1 enjoy Carathéodory regularity, i.e. these functions are measurable in x and continuous in $\boldsymbol{a}$. Moreover, there exists a constant $C_f > 0$ such that for a.a. $x \in \Omega_{\mathrm{act}}$ and for all $\boldsymbol{a} \in \mathbb{R}^3_{\boldsymbol{a}}$ we have the estimate

$$\big|f_0(x, \boldsymbol{a})\big| + \big|f_1(x, \boldsymbol{a})\big| \le C_f. \tag{4.39}$$

(iii) There exist L_0, $L_1 > 0$ such that for all $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathbb{R}^3_{\boldsymbol{a}}$ and for a.a. $x \in \Omega_{\mathrm{act}}$ we have

$$\big|f_0(x, \boldsymbol{a}_1) - f_0(x, \boldsymbol{a}_2)\big| \le L_0\, |\boldsymbol{a}_1 - \boldsymbol{a}_2|, \tag{4.40a}$$

$$\big|f_1(x, \boldsymbol{a}_1) - f_1(x, \boldsymbol{a}_2)\big| \le L_1\, |\boldsymbol{a}_1 - \boldsymbol{a}_2|. \tag{4.40b}$$

(iv) *There exists $C_b > 0$ such that for a.a. $x \in \Omega_{\mathrm{act}}$, for all $\mathbf{g_E} \in \mathbb{R}^3_{\boldsymbol{a}}$ and for all $\boldsymbol{a} \in \mathbb{R}^3_{\boldsymbol{a}}$ with $|\boldsymbol{a}| \geq C_b$ we have*
$$f(x, \boldsymbol{a}, \mathbf{g_E}) \cdot \boldsymbol{a} \leq 0. \tag{4.41}$$

The main difference of our setting to the situation studied in [Dum05] is the growth condition (4.41). For the analysis, this is only a minor difference, but from the authors point of view, our condition is an important generalization. For $C_b = 1/2$, our condition exactly forbids unphysical evolutions that do not leave the set $\mathcal{A}_0$ invariant, see Proposition 4.1.10 (iii). We also note that in [DuS12] the growth condition $f(x, \boldsymbol{a}, \mathbf{g_E}) \cdot \boldsymbol{a} \leq K|\boldsymbol{a}|^2$ for some $K > 0$ was imposed.

Remark 4.1.6. *From (i)–(iii) of the above hypothesis we can infer the following.*

(i) *For all $\boldsymbol{a}$, $\mathbf{g_{E1}}$, $\mathbf{g_{E2}} \in \mathbb{R}^3_{\boldsymbol{a}}$ and for a.a. $x \in \Omega_{\mathrm{act}}$ it holds*
$$\big|f_1(x, \boldsymbol{a})\mathbf{g_{E1}} - f_1(x, \boldsymbol{a})\mathbf{g_{E2}}\big| \leq C_f|\mathbf{g_{E1}} - \mathbf{g_{E2}}|. \tag{4.42}$$

(ii) *With $C_g := \max\{L_0, L_1, C_f\}$ we have that for all $\boldsymbol{a}$, $\mathbf{g_E} \in \mathbb{R}^3_{\boldsymbol{a}}$ and for a.a. $x \in \Omega_{\mathrm{act}}$ it holds*
$$\big|f(x, \boldsymbol{a}, \mathbf{g_E})\big| \leq C_g\big(1 + |\boldsymbol{a}|\big)\big(1 + |\mathbf{g_E}|\big). \tag{4.43}$$

We emphasize that the function F_δ from (4.36) in contrast to the function F from (4.21) actually satisfies this hypothesis. We particularly have

Remark 4.1.7. *For the concrete function F_δ from* (4.36) *the Lipschitz constants L_0 and L_1 depend on the regularization parameter $0 < \delta \ll 1/2$. Clearly, the function* F *from* (4.21) *satisfies all conditions* (4.39)–(4.43) *except* (4.40). *For the function* F, *we can infer* (4.43) *from the uniform boundedness of $\mathsf{C}_{\boldsymbol{a}}$ with respect to $\boldsymbol{a}$.*

In the following, we will usually neglect the x-dependence of f and write $f(\boldsymbol{a}, \mathsf{G}^*\mathbf{E})$ instead of $f(x, \boldsymbol{a}, \mathsf{G}^*\mathbf{E})$. In the context of a function f satisfying Hypothesis 4.1.5, we take our initial data from the set
$$\mathcal{IC}_{C_b} := \Big\{\big(\mathbf{E}_0, \mathbf{H}_0, \boldsymbol{a}_0\big) \in \mathbf{L}_{\mathrm{div}} : \ \boldsymbol{a}_0 \in \mathcal{A}_{C_b}\Big\}, \tag{4.44}$$
where the set $\mathcal{A}_{C_b}$ is defined by
$$\mathcal{A}_{C_b} := \Big\{\boldsymbol{a}_0 \in L^\infty(\Omega_{\mathrm{act}}) : \|\boldsymbol{a}_0\|_{L^\infty(\Omega_{\mathrm{act}})} \leq C_b\Big\}. \tag{4.45}$$

4.1.3. Notion of Solutions and A-priori Statements

In this subsection, we formulate our main problem of this chapter. We introduce our notion of solutions and state some a-priori estimates on possible solutions.

We are interested in the following problem where P is a placeholder for either F, F_δ or a function $f \in \mathcal{F}_{\mathrm{hyp}}$.

Problem 4.1.8. *Let a function* $\mathrm{P} \in \mathcal{F}_{\mathrm{hyp}} \cup \{\mathrm{F}, \mathrm{F}_\delta\}$ *be given. For* $\mathrm{P} \in \mathcal{F}_{\mathrm{hyp}}$ *assume that Condition 4.1.1 is satisfied and for* $\mathrm{P} \in \{\mathrm{F}, \mathrm{F}_\delta\}$ *assume that* (4.11) *also holds. Moreover, let initial data* $\big(\mathbf{E}_0, \mathbf{H}_0, \boldsymbol{a}_0\big) \in \mathcal{IC}_{C_b}$*, resp.* $\big(\mathbf{E}_0, \mathbf{H}_0, \boldsymbol{a}_0\big) \in \mathcal{IC}$ *be given. Find a triplet of functions* $\big(\mathbf{E}, \mathbf{H}, \boldsymbol{a}\big)$ *depending on* (x, t) *that satisfies*

$$\partial_t \mathbf{E} = \epsilon_0^{-1} \operatorname{curl} \mathbf{H} - \epsilon_0^{-1} \mathsf{G}\mathrm{P}(x, \boldsymbol{a}, \mathsf{G}^* \mathbf{E}) \qquad \text{in } [0, T] \times \mathbb{R}^3 \tag{4.46a}$$

$$\partial_t \mathbf{H} = -\mu_0^{-1} \operatorname{curl} \mathbf{E} \qquad \text{in } [0, T] \times \mathbb{R}^3 \tag{4.46b}$$

$$\partial_t \boldsymbol{a} = \mathrm{P}(x, \boldsymbol{a}, \mathsf{G}^* \mathbf{E}) \qquad \text{in } [0, T] \times \Omega_{\mathrm{act}} \tag{4.46c}$$

$$\big(\mathbf{E}(0),\, \mathbf{H}(0)\big) = \big(\mathbf{E}_0, \mathbf{H}_0\big) \quad \text{in } \mathbb{R}^3 \qquad \boldsymbol{a}(0) = \boldsymbol{a}_0 \quad \text{in } \Omega_{\mathrm{act}}. \tag{4.46d}$$

We note that for arbitrary functions $f \in \mathcal{F}_{\mathrm{hyp}}$, the above system of equations (4.46) can in general not be derived from a damped Hamiltonian system of the form (2.3).

In order to give a more concise notion of solutions, we use the notation from Jochmann[3] and introduce the operator[4] B and its formally adjoint B^* given by

$$\mathsf{B} := \begin{pmatrix} 0 & \epsilon_0^{-1} \operatorname{curl} \\ -\mu_0^{-1} \operatorname{curl} & 0 \end{pmatrix}, \qquad \mathsf{B}^* := \begin{pmatrix} 0 & -\mu_0^{-1} \operatorname{curl} \\ \epsilon_0^{-1} \operatorname{curl} & 0 \end{pmatrix}. \tag{4.47}$$

Next, we give the notion of solutions we want to consider. In this definition we encode that charge and current conservation hold, i.e. that the condition (4.17) is satisfied for all times $t \in [0.T]$.

Definition 4.1.9. *We call a triplet of functions* $\big(\mathbf{E}, \mathbf{H}, \boldsymbol{a}\big) \in C^0([0, T]; \mathbf{L}_{\mathrm{div}})$ *a weak solution to Problem 4.1.8 if the following holds.*

(i) For all $t \in [0, T]$ *and for a.a.* $x \in \Omega_{\mathrm{act}}$ *we have*

$$\boldsymbol{a}(x, t) = \boldsymbol{a}_0(x) + \int_0^t \mathrm{P}\big(x, \boldsymbol{a}(x, s), \mathsf{G}^* \mathbf{E}(x, s)\big)\, ds. \tag{4.48}$$

(ii) For all $\psi \in C_c^\infty([0, T) \times \mathbb{R}^3; \mathbb{R}^6)$ *we have*

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^3} \partial_t \psi(x, s) \cdot \big(\mathbf{E}(x, s), \mathbf{H}(x, s)\big)\, dx\, ds + \int_{\mathbb{R}^3} \psi(x, 0) \cdot \big(\mathbf{E}_0(x), \mathbf{H}_0(x)\big)\, dx \\ &- \int_0^t \int_{\mathbb{R}^3} \mathsf{B}^*[\psi(x, s)] \cdot \big(\mathbf{E}(x, s), \mathbf{H}(x, s)\big)\, dx\, ds \\ &= \int_0^t \int_{\mathbb{R}^3} \psi(x, s) \cdot \Big(\epsilon_0^{-1} \mathsf{G}\mathrm{P}\big(x, \boldsymbol{a}(x, s), \mathsf{G}^* \mathbf{E}(x, s)\big),\, 0\Big)\, dx\, ds. \end{aligned} \tag{4.49}$$

[3] See for example [Joc02b, Sec. 2].

[4] It should be possible, to replace the concrete operator B with any operator satisfying the condition stated in [Joc00] and performing the whole analysis for such an operator.

The next proposition contains some a-priori estimates and regularity properties of possible solutions.

Proposition 4.1.10. *Let $\big(\mathbf{E}, \mathbf{H}, \boldsymbol{a}\big)$ be a weak solution to Problem 4.1.8 in the sense of Definition 4.1.9. Then it holds*

(i) For all $t \in [0,T]$, the pair $\big(\mathbf{E}, \mathbf{H}\big)$ satisfies the energy balance

$$\big\|\big(\epsilon_0^{1/2}\,\mathbf{E}(t), \mu_0^{1/2}\,\mathbf{H}(t)\big)\big\|^2_{L^2(\mathbb{R}^3)} = \big\|\big(\epsilon_0^{1/2}\,\mathbf{E}_0, \mu_0^{1/2}\,\mathbf{H}_0\big)\big\|^2_{L^2(\mathbb{R}^3)} - 2\int_0^t \int_{\Omega_{\text{act}}} \mathsf{G}^*\mathbf{E} \cdot \mathrm{P}\big(\boldsymbol{a}, \mathsf{G}^*\mathbf{E}\big)\, dx\, ds. \tag{4.50}$$

(ii) We have $\boldsymbol{a} \in W^{1,\infty}((0,T); L^2(\Omega_{\text{act}}))$ and equation (4.46c) is satisfied for almost all $(x,t) \in (0,T) \times \Omega_{\text{act}}$. In particular, for a.a. $x \in \Omega_{\text{act}}$ it holds $\boldsymbol{a}(x,0) = \boldsymbol{a}_0(x)$.

(iii) The function $\boldsymbol{a}$ satisfies

$$\forall\, t \in [0,T]: \qquad \|\boldsymbol{a}(t)\|_{L^\infty(\Omega_{\text{act}})} \le C_b. \tag{4.51}$$

Thus, every weak solution $\boldsymbol{a}$ to (4.46c) satisfies $\boldsymbol{a}(t) \in \mathcal{A}_{C_b}$ for all $t \in [0,T]$.[5] Furthermore, for all $p \in [2,\infty)$ it holds $\boldsymbol{a} \in C^0([0,T]; L^p(\Omega_{\text{act}}))$.

(iv) In the case $\mathrm{P} = \mathrm{F}$, the free energy functional $\widehat{\mathcal{F}}_{\vartheta_*}$ from (4.13) is absolutely continuous in time t. Moreover, introducing $\mathbf{U} := \big(\mathbf{E}, \mathbf{H}, \boldsymbol{a}\big)$, the functional $\widehat{\mathcal{F}}_{\vartheta_*}$ is a Liapunov function in the sense that the following estimates hold

$$\forall_{\text{a.a.}}\, t \in [0,T]: \quad \frac{d}{dt}\widehat{\mathcal{F}}_{\vartheta_*}\big(\mathbf{U}(t)\big) = -\frac{1}{\vartheta_*}\Big\langle D\widehat{\mathcal{F}}_{\vartheta_*}\big(\mathbf{U}\big),\, \widehat{\mathbb{K}}\big(\mathbf{U}\big)\, D\widehat{\mathcal{F}}_{\vartheta_*}\big(\mathbf{U}\big)\Big\rangle \le 0, \tag{4.52}$$

$$\forall\, t \in [0,T]: \quad \widehat{\mathcal{F}}_{\vartheta_*}\big(\mathbf{U}(t)\big) + \frac{1}{\vartheta_*}\int_0^t \Big\langle D\widehat{\mathcal{F}}_{\vartheta_*}\big(\mathbf{U}(s)\big),\, \widehat{\mathbb{K}}\big(\mathbf{U}(s)\big)\, D\widehat{\mathcal{F}}_{\vartheta_*}\big(\mathbf{U}(s)\big)\Big\rangle\, ds = \widehat{\mathcal{F}}_{\vartheta_*}\big(\mathbf{U}_0\big). \tag{4.53}$$

Here, $\big\langle \cdot, \cdot \big\rangle$ denotes the dual pairing in $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\Omega_{\text{act}})$. This means for $\mathrm{P} = \mathrm{F}$ the statements of Proposition 2.1.5 hold true for every weak solution $\big(\mathbf{E}, \mathbf{H}, \boldsymbol{a}\big)$ to Problem 4.1.8.

(v) The statement (iv) literally holds by replacing $\widehat{\mathcal{F}}_{\vartheta_*}$, $\widehat{\mathbb{K}}$ and F with their regularized versions $\widehat{\mathcal{F}}^{\delta}_{\vartheta_*}$, $\widehat{\mathbb{K}}_\delta$ and F_δ.

[5] In the case $\mathrm{P} \in \{\mathrm{F}, \mathrm{F}_\delta\}$ we have $C_b = 1/2$.

Proof. (i) If $(\mathbf{E}, \mathbf{H}, \boldsymbol{a})$ is a weak solution to Problem 4.1.8 in the sense of Definition 4.1.9, then, the pair $(\underline{\mathbf{E}}, \underline{\mathbf{H}}) = (\epsilon_0^{1/2}\,\mathbf{E}, \mu_0^{1/2}\,\mathbf{H})$ is the unique solution to the Cauchy problem for the symmetric hyperbolic system

$$\partial_t(\underline{\mathbf{E}}, \underline{\mathbf{H}}) - \underline{\mathsf{B}}(\underline{\mathbf{E}}, \underline{\mathbf{H}}) = -\big(\epsilon_0^{-1}\mathsf{G}\mathrm{P}(\boldsymbol{a}, \mathsf{G}^*\underline{\mathbf{E}}),\, 0\big), \qquad \underline{\mathsf{B}} := (\epsilon_0\mu_0)^{-1/2}\begin{pmatrix} 0 & \operatorname{curl} \\ -\operatorname{curl} & 0 \end{pmatrix} \tag{4.54}$$

with initial data $(\underline{\mathbf{E}}(0), \underline{\mathbf{H}}(0)) = (\epsilon_0^{1/2}\,\mathbf{E}_0, \mu_0^{1/2}\,\mathbf{H}_0)$. For the right hand side we have

$$\big(\epsilon_0^{-1}\mathsf{G}\mathrm{P}(\boldsymbol{a}, \mathsf{G}^*\underline{\mathbf{E}}),\, 0\big) \in L^2((0,T), L^2(\mathbb{R}^3)). \tag{4.55}$$

Therefore, taking into account the linearity of P in $\mathbf{E}$, Proposition A.5.3 yields (4.50) by reinserting $\epsilon_0^{1/2}\,\mathbf{E}$ for $\underline{\mathbf{E}}$ since G is the adjoint of G^*.

(ii) Due to our assumptions on the data and the pointwise estimate (4.23), it is clear that the integrand in (4.48) belongs to the space $L^\infty((0,T); L^2(\Omega_{\text{act}}))$, i.e. the function $\partial_t\boldsymbol{a}$ is well defined in the space $L^\infty((0,T); L^2(\Omega_{\text{act}}))$. Thus, we have $\boldsymbol{a} \in W^{1,\infty}((0,T); L^2(\Omega_{\text{act}}))$. In particular, $\partial_t\boldsymbol{a}$ satisfies (4.46c). From (4.48) it is also clear, that we have $\boldsymbol{a}(x,0) = \boldsymbol{a}_0(x)$ for a.a. $x \in \Omega_{\text{act}}$.

(iii) First, we show the a-priori bound for almost all $t \in [0,T]$, i.e. we show that we have $\|\boldsymbol{a}\|_{L^\infty((0,T);L^\infty(\Omega_{\text{act}}))} \le C_b$. To this end, we assume that there exists a set $G \subset (0,T)\times\Omega_{\text{act}}$ of positive measure such that $|\boldsymbol{a}(x,t)| > C_b$ in G. Next, we define

$$g(x,t) := \max\Big\{\big|\boldsymbol{a}(x,t)\big| - C_b,\, 0\Big\}, \qquad (x,t) \in \Omega_T. \tag{4.56}$$

For a.a. $x \in \Omega_{\text{act}}$ and for all $t \in [0,T]$, the function g satisfies

$$\frac{1}{2}\frac{d}{dt}\big|g(x,t)\big|^2 = \frac{g(x,t)}{|\boldsymbol{a}(x,t)|}\,\partial_t\boldsymbol{a}(x,t)\cdot\boldsymbol{a}(x,t). \tag{4.57}$$

Integration over $(0,t)\times\Omega_{\text{act}}$ for some arbitrary $t \in [0,T]$ yields

$$\int_{\Omega_{\text{act}}} \frac{1}{2}\big|g(x,t)\big|^2\,dx - \int_0^t \int_{\Omega_{\text{act}}} \frac{g(x,s)}{|\boldsymbol{a}(x,s)|}\,\mathrm{P}(x, \boldsymbol{a}, \mathsf{G}^*\mathbf{E})\cdot\boldsymbol{a}(x,s)\,ds\,dx = \int_{\Omega_{\text{act}}} \frac{1}{2}\big|g(x,0)\big|^2\,dx.$$

The right hand side of this equation is obviously equal to zero due to $\boldsymbol{a}_0 \in \mathcal{A}_{C_b}$. Furthermore, we have $|g(x,t)|^2 \ge 0$. Thus, the first term on the left hand side is non-negative. For the remaining term note that on the one hand, we only get a non-zero contribution for those (x,t) satisfying $|\boldsymbol{a}(x,t)| > C_b$ due to the definition of the function g. On the other hand, these (x,t) yield $\mathrm{P}(x, \boldsymbol{a}(x,t), \mathsf{G}^*\mathbf{E}(x,t))\cdot\boldsymbol{a}(x,t) \le 0$, see (4.41) or (4.22). Therefore, we have

$$-\int_0^t \int_{\Omega_{\text{act}}} \frac{g(x,s)}{|\boldsymbol{a}(x,s)|}\,\mathrm{P}(x, \boldsymbol{a}, \mathsf{G}^*\mathbf{E})\cdot\boldsymbol{a}(x,s)\,ds\,dx \ge 0. \tag{4.58}$$

This implies $g(x,t) = 0$ for a.a. $(x,t) \in (0,T)\times\Omega_{\text{act}}$. Hence, such a set G cannot exist.

Next, we show that the a-priori bound holds for all $t \in [0,T]$. Due to (ii) of the present proposition and the embedding[6] $W^{1,\infty}((0,T);L^2(\Omega_{\text{act}})) \hookrightarrow C^0([0,T];L^2(\Omega_{\text{act}}))$ we have that for all t, $\{t_n\}_{n\in\mathbb{N}} \subset [0,T]$ with $t_n \to t$ it holds

$$\|\boldsymbol{a}(t_n) - \boldsymbol{a}(t)\|_{L^2(\Omega_{\text{act}})} \longrightarrow 0 \qquad \text{as } n \to \infty. \tag{4.59}$$

Furthermore, we have $\|\boldsymbol{a}\|_{L^\infty((0,T);L^\infty(\Omega_{\text{act}}))} \leq C_b$. Thus, there exists some set $N \subset [0,T]$ of measure zero such that for all $t \in [0,T] \setminus N$ it holds $\|\boldsymbol{a}(t)\|_{L^\infty(\Omega_{\text{act}})} \leq C_b$. For arbitrary $\widetilde{t} \in N$ let some sequence $\{t_n\}_{n\in\mathbb{N}} \subset [0,T] \setminus N$ be given with $t_n \to \widetilde{t}$. Then, it holds $\|\boldsymbol{a}(t_n)\|_{L^\infty(\Omega_{\text{act}})} \leq C_b$ for all $n \in \mathbb{N}$. This implies the existence of some subsequence $\{t_{n_k}\}_{k\in\mathbb{N}}$ and some $\widetilde{\boldsymbol{a}} \in L^\infty(\Omega_{\text{act}})$ with

$$\boldsymbol{a}(t_{n_k}) \longrightarrow \widetilde{\boldsymbol{a}} \qquad \text{weakly-}* \text{ in } L^\infty(\Omega_{\text{act}}) \quad \text{as } k \to \infty. \tag{4.60}$$

Due to the uniqueness of the limit, we may infer the equality $\widetilde{\boldsymbol{a}} = \boldsymbol{a}(\widetilde{t})$ from (4.59) and due to the weak-$*$-lower semi-continuity of the norm we may infer $\|\boldsymbol{a}(\widetilde{t})\|_{L^\infty(\Omega_{\text{act}})} \leq C_b$. The arbitrariness of $\widetilde{t} \in N$ yields the assertion.

Concluding, we show the regularity statement. For arbitrary $p \in [2,\infty)$ and $t \in (0,T)$ we have

$$\begin{aligned}
\lim_{\vartheta\to 0}\|\boldsymbol{a}(t+\vartheta) - \boldsymbol{a}(t)\|^p_{L^p(\Omega_{\text{act}})} &= \lim_{\vartheta\to 0}\int_{\Omega_{\text{act}}} |\boldsymbol{a}(t+\vartheta) - \boldsymbol{a}(t)|^{p-2} \cdot |\boldsymbol{a}(t+h) - \boldsymbol{a}(t)|^2 \, dx \\
&\leq \lim_{\vartheta\to 0} \|\boldsymbol{a}(t+\vartheta) - \boldsymbol{a}(t)\|^{p-2}_{L^\infty(\Omega_{\text{act}})} \cdot \|\boldsymbol{a}(t+h) - \boldsymbol{a}(t)\|^2_{L^2(\Omega_{\text{act}})} \\
&\leq \lim_{\vartheta\to 0} \big(\|\boldsymbol{a}(t+\vartheta)\|_{L^\infty(\Omega_{\text{act}})} + \|\boldsymbol{a}(t)\|_{L^\infty(\Omega_{\text{act}})}\big)^{p-2} \cdot \|\boldsymbol{a}(t+\vartheta) - \boldsymbol{a}(t)\|^2_{L^2(\Omega_{\text{act}})} = 0.
\end{aligned}$$

Hence, we have shown $\boldsymbol{a} \in C^0([0,T];L^p(\Omega_{\text{act}}))$ for all $p \in [2,\infty)$.

(iv) It is clear that (4.53) follows from (4.52) by integrating over some interval $(0,t) \subset [0,T]$. Since curl is an unbounded operator on $L^2(\mathbb{R}^3)$, we cannot argue on the formal level as in Chapter 2. Instead, we prove this claim "by foot". On the one hand, in view of (4.46c), (2.84) and (i) of the present proposition, we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}\big(\mathbf{E},\mathbf{H},\boldsymbol{a}\big) &= \frac{d}{dt}\bigg(\int_{\mathbb{R}^3} \frac{\epsilon_0}{2}|\mathbf{E}|^2 + \frac{\mu_0}{2}|\mathbf{H}|^2 \, dx + 2\int_{\Omega_{\text{act}}} \mathbf{h}\cdot\boldsymbol{a} + k_B\vartheta_*\sigma(|\boldsymbol{a}|)\, dx\bigg) \\
&= -\int_{\Omega_{\text{act}}} \mathsf{G}^*\mathbf{E}\cdot \mathrm{F}\big(\boldsymbol{a},\mathsf{G}^*\mathbf{E}\big)\, dx + 2\int_{\Omega_{\text{act}}} \Big(\mathbf{h} + k_B\vartheta_*\frac{\boldsymbol{a}}{\lambda(|\boldsymbol{a}|)}\Big)\cdot \mathrm{F}(\boldsymbol{a},\mathsf{G}^*\mathbf{E})\, dx \\
&= -\int_{\Omega_{\text{act}}} \Big(\mathsf{G}^*\mathbf{E} - 2\mathbf{h} - 2\,k_B\vartheta_*\frac{\boldsymbol{a}}{\lambda(|\boldsymbol{a}|)}\Big)\cdot \mathrm{F}(\boldsymbol{a},\mathsf{G}^*\mathbf{E})\, dx.
\end{aligned} \tag{4.61}$$

[6]See [Zei90, Problem 23.13a, p. 450].

Obviously, due to $\boldsymbol{a} \times \boldsymbol{a} = 0$ we can write $\mathrm{F}(\boldsymbol{a}, \mathsf{G}^*\mathbf{E})$ in the following way

$$\mathrm{F}(x, \boldsymbol{a}, \mathsf{G}^*\mathbf{E}) = \frac{1}{\hbar}\boldsymbol{a} \times \Big(2\mathbf{h} + k_B\vartheta_* \frac{\boldsymbol{a}}{\lambda(|\boldsymbol{a}|)} - \mathsf{G}^*\mathbf{E}\Big) \\ + \frac{2}{\vartheta_*}\sum_{m=1}^{M} \mathbf{q}_m \times \mathsf{C}_{\boldsymbol{a}}\Big(\mathbf{q}_m \times \Big(2\mathbf{h} + 2\,k_B\vartheta_* \frac{\boldsymbol{a}}{\lambda(|\boldsymbol{a}|)} - \mathsf{G}^*\mathbf{E}\Big)\Big). \tag{4.62}$$

Therefore, introducing $\mathbf{V} := \Big(2\mathbf{h} + 2\,k_B\vartheta_* \frac{\boldsymbol{a}}{\lambda(|\boldsymbol{a}|)} - \mathsf{G}^*\mathbf{E}\Big)$, we get from (4.61) the equality

$$\begin{aligned} \frac{d}{dt}\mathcal{F}\big(\mathbf{E}, \mathbf{H}, \boldsymbol{a}\big) &= \frac{2}{\vartheta_*}\sum_{m=1}^{M}\int_{\Omega_{\mathrm{act}}} \mathbf{V} \cdot \Big(\mathbf{q}_m \times \mathsf{C}_{\boldsymbol{a}}\big(\mathbf{q}_m \times \mathbf{V}\big)\Big)\,dx \\ &= -\frac{2}{\vartheta_*}\sum_{m=1}^{M}\int_{\Omega_{\mathrm{act}}} \big(\mathbf{q}_m \times \mathbf{V}\big) \cdot \mathsf{C}_{\boldsymbol{a}}\big(\mathbf{q}_m \times \mathbf{V}\big)\,dx \end{aligned} \tag{4.63}$$

due to the symmetry of $\mathsf{C}_{\boldsymbol{a}}$. The positivity of $\mathsf{C}_{\boldsymbol{a}}$ proved in Lemma A.2.3 yields

$$\frac{d}{dt}\mathcal{F}(\mathbf{E}, \mathbf{H}, \boldsymbol{a}) \leq 0. \tag{4.64}$$

On the other hand, we have

$$\widehat{\mathbb{K}}\big(\mathbf{E}, \mathbf{H}, \boldsymbol{a}\big)\, D\widehat{\mathcal{F}}_{\vartheta_*}\big(\mathbf{E}, \mathbf{H}, \boldsymbol{a}\big) = \epsilon_0^{-1}\begin{pmatrix} \mathsf{G}\widehat{\mathcal{K}}(\boldsymbol{a})\big[\mathsf{G}^*\mathbf{E} - 2\mathbf{h} - 2\vartheta_* k_B \frac{\boldsymbol{a}}{\lambda(|\boldsymbol{a}|)}\big] \\ 0 \\ \epsilon_0\widehat{\mathcal{K}}(\boldsymbol{a})\big[-\mathsf{G}^*\mathbf{E} + 2\mathbf{h} + 2\vartheta_* k_B \frac{\boldsymbol{a}}{\lambda(|\boldsymbol{a}|)}\big] \end{pmatrix}, \tag{4.65}$$

thus,

$$\begin{aligned} &\Big\langle D\widehat{\mathcal{F}}_{\vartheta_*}\big(\mathbf{E}, \mathbf{H}, \boldsymbol{a}\big), \widehat{\mathbb{K}}\big(\mathbf{E}, \mathbf{H}, \boldsymbol{a}\big)\, D\widehat{\mathcal{F}}_{\vartheta_*}\big(\mathbf{E}, \mathbf{H}, \boldsymbol{a}\big)\Big\rangle \\ &\quad = -\int_{\Omega_{\mathrm{act}}} \Big(\mathsf{G}^*\mathbf{E} - 2\mathbf{h} - 2\vartheta_* k_B \frac{\boldsymbol{a}}{\lambda(|\boldsymbol{a}|)}\Big) \cdot \widehat{\mathcal{K}}(\boldsymbol{a})\Big[\mathsf{G}^*\mathbf{E} - 2\mathbf{h} - 2\vartheta_* k_B \frac{\boldsymbol{a}}{\lambda(|\boldsymbol{a}|)}\Big]\,dx. \end{aligned} \tag{4.66}$$

Recalling the definition of $\widehat{\mathcal{K}}(\boldsymbol{a})$ from (2.81), a comparison with (4.63) yields

$$\frac{d}{dt}\mathcal{F}(\mathbf{E}, \mathbf{H}, \boldsymbol{a}) = -\frac{1}{\vartheta_*}\Big\langle D\widehat{\mathcal{F}}_{\vartheta_*}\big(\mathbf{E}, \mathbf{H}, \boldsymbol{a}\big), \widehat{\mathbb{K}}\big(\mathbf{E}, \mathbf{H}, \boldsymbol{a}\big)\, D\widehat{\mathcal{F}}_{\vartheta_*}\big(\mathbf{E}, \mathbf{H}, \boldsymbol{a}\big)\Big\rangle \tag{4.67}$$

and finishes our proof. Statement (v) is proved exactly as statement (iv). □

4.2. Results

For simplicity, from now on we assume

$$\epsilon_0 = 1 \qquad \text{and} \qquad \mu_0 = 1 \tag{4.68}$$

so that for every given function $\boldsymbol{a} \in L^2((0,T); L^2(\Omega_{\text{act}}))$ system (4.70a)–(4.70b) is symmetric hyperbolic in the sense of Definition A.5.1. This assumption is not necessary for our analysis, since we could instead consider the equivalent system one gets by replacing $(\mathbf{E}, \mathbf{H})$ with $(\epsilon_0^{1/2}\,\mathbf{E}, \mu_0^{1/2}\,\mathbf{H})$ and the differential operator B with the symmetric operator $\underline{\mathsf{B}}$ from (4.54). We perform the above assumption in order to simplify our notation, though. In particular, this assumption implies that the differential operator B and its adjoint B^* are symmetric now and given by

$$\mathsf{B} := \begin{pmatrix} 0 & \operatorname{curl} \\ -\operatorname{curl} & 0 \end{pmatrix}, \qquad \mathsf{B}^* := \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl} & 0 \end{pmatrix}. \tag{4.69}$$

We have the following result concerning the existence of solutions to Problem 4.1.8.

Theorem 4.2.1. *Let a bounded open set $\Omega_{\text{act}} \subset \mathbb{R}^3$, a time $T > 0$, a function $f : \Omega_{\text{act}} \times \mathbb{R}^3_{\boldsymbol{a}} \times \mathbb{R}^3_{\boldsymbol{a}} \longrightarrow \mathbb{R}^3_{\boldsymbol{a}}$ satisfying the assumptions of Hypothesis 4.1.5 as well as initial data $(\mathbf{E}_0, \mathbf{H}_0, \boldsymbol{a}_0) \in \mathcal{IC}_{C_b}$ be given. Furthermore, let a function $\gamma^* \in L^\infty(\Omega_{\text{act}}; \mathcal{L}(\mathbb{R}^3_{\boldsymbol{a}}, \mathbb{R}^3_{\mathbf{E}}))$ with corresponding dipole moment operator G^* according to (4.7) be given and let G denote the adjoint of G^* given by (4.8).*
Then, there exists a weak solution $(\mathbf{E}, \mathbf{H}, \boldsymbol{a}) \in C^0([0,T]; \mathbf{L}_{\text{div}})$ to the following Cauchy problem in the sense of Definition 4.1.9

$$\partial_t \mathbf{E} = \operatorname{curl} \mathbf{H} - \mathsf{G} f(\boldsymbol{a}, \mathsf{G}^* \mathbf{E}) \qquad \textit{in } [0,T] \times \mathbb{R}^3 \tag{4.70a}$$

$$\partial_t \mathbf{H} = -\operatorname{curl} \mathbf{E} \qquad \textit{in } [0,T] \times \mathbb{R}^3 \tag{4.70b}$$

$$\partial_t \boldsymbol{a} = f(\boldsymbol{a}, \mathsf{G}^* \mathbf{E}) \qquad \textit{in } [0,T] \times \Omega_{\text{act}} \tag{4.70c}$$

$$\big(\mathbf{E}(0),\, \mathbf{H}(0)\big) = \big(\mathbf{E}_0, \mathbf{H}_0\big) \quad \textit{in } \mathbb{R}^3 \qquad \boldsymbol{a}(0) = \boldsymbol{a}_0 \quad \textit{in } \Omega_{\text{act}}. \tag{4.70d}$$

Moreover, for all $t \in [0,T]$ it holds $\boldsymbol{a}(t) \in \mathcal{A}_{C_b}$ and the triplet $(\mathbf{E}, \mathbf{H}, \boldsymbol{a})$ satisfies the statements of Proposition 4.1.10.

Since for all $0 < \delta \ll 1/2$ we have $\mathrm{F}_\delta \in \mathcal{F}_{\text{hyp}}$, the above theorem implies the following result.

Theorem 4.2.2. *Let a regularization parameter $\delta \in [0, 1/2]$ with $\delta \ll 1/2$, a bounded open set $\Omega_{\text{act}} \subset \mathbb{R}^3$, a time $T > 0$, functions $\mathbf{h}, \mathbf{q}_m \in L^\infty(\Omega_{\text{act}}; \mathbb{R}^3_{\boldsymbol{a}})$ as well as initial data $(\mathbf{E}_0, \mathbf{H}_0, \boldsymbol{a}_0) \in \mathcal{IC}$ be given. Furthermore, let a function $\gamma^* \in L^\infty(\Omega_{\text{act}}; \mathcal{L}(\mathbb{R}^3_{\boldsymbol{a}}, \mathbb{R}^3_{\mathbf{E}}))$ with corresponding dipole moment operator G^* according to (4.7) be given and let G denote the adjoint of G^* given by (4.8).*

Then, there exists a weak solution $\big(\mathbf{E},\mathbf{H},\boldsymbol{a}\big) \in C^0([0,T];\mathbf{L}_{\mathrm{div}})$ *to the following Cauchy problem in the sense of Definition 4.1.9*

$$\partial_t\mathbf{E} = \operatorname{curl}\mathbf{H} - \mathsf{G}\mathrm{F}_\delta(x,\boldsymbol{a},\mathsf{G}^*\mathbf{E}) \qquad \text{in } [0,T]\times\mathbb{R}^3 \tag{4.71a}$$

$$\partial_t\mathbf{H} = -\operatorname{curl}\mathbf{E} \qquad \text{in } [0,T]\times\mathbb{R}^3 \tag{4.71b}$$

$$\partial_t\mathbf{a} = \mathrm{F}_\delta(x,\boldsymbol{a},\mathsf{G}^*\mathbf{E}) \qquad \text{in } [0,T]\times\Omega_{\mathrm{act}} \tag{4.71c}$$

$$\big(\mathbf{E}(0),\,\mathbf{H}(0)\big) = \big(\mathbf{E}_0,\mathbf{H}_0\big) \quad \text{in } \mathbb{R}^3 \qquad \boldsymbol{a}(0) = \boldsymbol{a}_0 \quad \text{in } \Omega_{\mathrm{act}}. \tag{4.71d}$$

Moreover, for all $t \in [0,T]$ *it holds* $\boldsymbol{a}(t) \in \mathcal{A}_0$ *and the triplet* $\big(\mathbf{E},\mathbf{H},\boldsymbol{a}\big)$ *satisfies the statements of Proposition 4.1.10.*

4.3. An Existence Proof

In this section we prove Theorem 4.2.1. A major part of our proof closely follows the lines of [JMR00a] (see also [Dum05]) and uses the main ideas developed there. First, we will give an approximation of Problem 4.1.8 with $\mathrm{P} \in \mathcal{F}_{\mathrm{hyp}}$ and prove existence and uniqueness for the solution of the approximating Cauchy problem. The rest of Section 4.3 is devoted to prove that, up to a subsequence, the series of approximating solutions converges to a weak solution of Problem 4.1.8 with $\mathrm{P} \in \mathcal{F}_{\mathrm{hyp}}$. The major step of the whole proof will be performed in Section 4.3.4.

4.3.1. Approximating Solutions

In this section, we state a Cauchy problem where on the one hand system (4.70a)–(4.70c) and on the other hand the Cauchy data (4.70d) is approximated. Moreover, we show that the approximating problem admits a unique smooth solution. Namely, for a fixed $T > 0$, given initial data $\big(\mathbf{E}_0,\mathbf{H}_0,\boldsymbol{a}_0\big) \in \mathbf{L}_{\mathrm{div}}$ with $\boldsymbol{a}_0$ satisfying $\boldsymbol{a}_0 \in \mathcal{A}_{C_b}$ and a smoothing Fourier multiplier operator S^λ, we define smooth Cauchy data

$$\big(\mathbf{E}_0^\lambda,\mathbf{H}_0^\lambda,\boldsymbol{a}_0^\lambda\big) := \big(\mathsf{S}^\lambda[\mathbf{E}_0],\mathsf{S}^\lambda[\mathbf{H}_0],\boldsymbol{a}_0\big) \tag{4.72}$$

and consider the Cauchy problem

$$\partial_t\mathbf{E} = \operatorname{curl}\mathbf{H} - \mathsf{S}^\lambda\big[\mathsf{G}f(\boldsymbol{a},\mathsf{G}^*\mathbf{E})\big] \qquad \text{in } [0,T]\times\mathbb{R}^3 \tag{4.73a}$$

$$\partial_t\mathbf{H} = -\operatorname{curl}\mathbf{E} \qquad \text{in } [0,T]\times\mathbb{R}^3 \tag{4.73b}$$

$$\partial_t\boldsymbol{a} = f(\boldsymbol{a},\mathsf{G}^*\mathbf{E}) \qquad \text{in } [0,T]\times\Omega_{\mathrm{act}} \tag{4.73c}$$

$$\big(\mathbf{E}(0),\,\mathbf{H}(0)\big) = \big(\mathbf{E}_0^\lambda,\mathbf{H}_0^\lambda\big) \quad \text{in } \mathbb{R}^3, \qquad \boldsymbol{a}(0) = \boldsymbol{a}_0^\lambda \quad \text{in } \Omega_{\mathrm{act}}. \tag{4.73d}$$

In contrast to the Problem 4.1.8, for $f = \mathrm{F}_\delta$ from (4.36) this system cannot be written in the form of a damped Hamiltonian system in the sense of Definition 2.1.1.

For the rest of this chapter we fix a cut-off function $\chi \in C_c^\infty(\mathbb{R}^3, [0,1])$ that satisfies Hypothesis A.3.9, i.e. $\chi(|x|) = 1$ if $|x| \leq 1/2$, $\chi(|x|) = 0$ if $|x| \geq 1$ and χ is even. Moreover, for every $\lambda \geq 1$, we define $\chi_\lambda(x) := \chi(x/\lambda)$. Denoting the Fourier transformation by[7] $\mathcal{F}$, we define the operator S^λ via[8]

$$\mathsf{S}^\lambda : \begin{cases} L^2(\mathbb{R}^3) \longrightarrow L^2_\lambda(\mathbb{R}^3) \\ \qquad \mathbf{U} \longmapsto \mathcal{F}^{-1}[\chi_\lambda \mathcal{F}\,\mathbf{U}]. \end{cases} \tag{4.74}$$

with

$$L^2_\lambda(\mathbb{R}^3) := \Big\{ \mathbf{U} \in L^2(\mathbb{R}^3) :\ \operatorname{supp}(\mathcal{F}\,\mathbf{U}) \subset \overline{B_\lambda(0)} \Big\} \tag{4.75}$$

and $\overline{B_\lambda(0)} := \{\xi \in \mathbb{R}^3 :\ |\xi| \leq \lambda\}$. The space $L^2_\lambda(\mathbb{R}^3)$ is endowed with the usual norm in $L^2(\mathbb{R}^3)$. We stress that the Fourier transformation (and its inverse) as well as the multiplication with χ_λ are to be done component wise. A comprehensive discussion of the operator S^λ is given in Section A.3. Since the projectors $\mathsf{P}_\parallel$ and $\mathsf{P}_\perp$ commute[9] with the smoothing operator S^λ, charge and current conservation (4.17) for the initial data yield the condition

$$\mathsf{P}_\parallel \mathbf{H}_0^\lambda = 0, \qquad \mathsf{P}_\parallel \big(\mathbf{E}_0^\lambda + \mathsf{S}^\lambda \mathsf{G} \boldsymbol{a}_0^\lambda \big) = 0 \tag{4.76}$$

for the smoothed initial data $\big(\mathbf{E}_0^\lambda, \mathbf{H}_0^\lambda, \boldsymbol{a}_0^\lambda\big)$.

In dimension $d = 3$, the differential operator curl can be expressed as a matrix-valued Fourier multiplier by

$$\mathcal{F}[\operatorname{curl} \mathbf{U}](\xi) = \pi_{\text{curl}}(\xi)\, \widehat{\mathbf{U}}(\xi), \qquad \text{with} \qquad \pi_{\text{curl}}(\xi) = 2\pi i \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}. \tag{4.77}$$

Therefore, it is clear that the differential operator curl maps the space $L^2_\lambda(\mathbb{R}^3)$ into itself. Due to the Bernstein type lemma (see Lemma A.3.10), for $\mathbf{U} \in L^2_\lambda(\mathbb{R}^3)$ it holds

$$\|\mathbf{U}\|_{L^\infty(\mathbb{R}^3)} \leq C_{\text{bsl}} \lambda^{3/2} \|\mathbf{U}\|_{L^2(\mathbb{R}^3)}. \tag{4.78}$$

Due to (ii) of Remark 4.1.6 and the definition of the operator G^* from (4.7) we have that for $\mathbf{E} \in L^2_\lambda(\mathbb{R}^3)$ and $\boldsymbol{a} \in (L^2 \cap L^\infty)(\Omega_{\text{act}})$ it holds

$$f(\boldsymbol{a}, \mathsf{G}^*\mathbf{E}) \in (L^2 \cap L^\infty)(\Omega_{\text{act}}) \qquad \text{and} \qquad \mathsf{S}^\lambda \mathsf{G} f(\boldsymbol{a}, \mathsf{G}^*\mathbf{E}) \in L^2_\lambda(\mathbb{R}^3). \tag{4.79}$$

Thus, the considerations above imply that the operator $\mathcal{T}^\lambda$ defined by

$$\mathcal{T}^\lambda : \big(\mathbf{E}, \mathbf{H}, \boldsymbol{a}\big) \longmapsto \Big(\Big(\operatorname{curl} \mathbf{H} - \mathsf{S}^\lambda \mathsf{G} f(\boldsymbol{a}, \mathsf{G}^*\mathbf{E}) \Big), \quad -\operatorname{curl} \mathbf{E}, \quad f(\boldsymbol{a}, \mathsf{G}^*\mathbf{E}) \Big) \tag{4.80}$$

[7] We understand that there should be no confusion with the free energy functional $\mathcal{F}$.

[8] For more details on the Fourier transformation and the smoothing operator S^λ we refer to Sec. A.3.

[9] Note that the smoothing operator S^λ commutes with all pure multiplier operators having matrix-valued symbols.

maps the Banach space $L^2_\lambda(\mathbb{R}^3)\times L^2_\lambda(\mathbb{R}^3)\times(L^2\cap L^\infty)(\Omega_{\text{act}})$ into itself. With the definition of the operator $\mathcal{T}^\lambda$, we can write system (4.70a)–(4.70c) as an ODE in the Banach space $L^2_\lambda(\mathbb{R}^3)\times L^2_\lambda(\mathbb{R}^3)\times(L^2\cap L^\infty)(\Omega_{\text{act}})$. Thus, the Cauchy problem (4.73) with initial data $\big(\mathbf{E}_0^\lambda, \mathbf{H}_0^\lambda, \boldsymbol{a}_0^\lambda\big) \in L^2_\lambda(\mathbb{R}^3)\times L^2_\lambda(\mathbb{R}^3)\times(L^2\cap L^\infty)(\Omega_{\text{act}})$ is equivalent to the initial value problem

$$\frac{d}{dt}\big(\mathbf{E},\mathbf{H},\boldsymbol{a}\big) = \mathcal{T}^\lambda\big(\mathbf{E},\mathbf{H},\boldsymbol{a}\big), \qquad \big(\mathbf{E}(0),\mathbf{H}(0),\boldsymbol{a}(0)\big) = \big(\mathbf{E}_0^\lambda,\mathbf{H}_0^\lambda,\boldsymbol{a}_0^\lambda\big) \tag{4.81}$$

in the Banach space $L^2_\lambda(\mathbb{R}^3)\times L^2_\lambda(\mathbb{R}^3)\times(L^2\cap L^\infty)(\Omega_{\text{act}})$. We have the following result concerning the regularized Cauchy problem (4.73).

Lemma 4.3.1. *For every $\lambda \geq 1$ and for every triplet*

$$\big(\mathbf{E}_0^\lambda,\mathbf{H}_0^\lambda,\boldsymbol{a}_0^\lambda\big) \in L^2_\lambda(\mathbb{R}^3)\times L^2_\lambda(\mathbb{R}^3)\times(L^2\cap L^\infty)(\Omega_{\text{act}}) \tag{4.82}$$

satisfying relations (4.76) *and $\boldsymbol{a}_0^\lambda \in \mathcal{A}_{C_b}$, there exists a maximal time $T>0$ such that the initial value problem* (4.81) *admits a unique (strong) solution*

$$\big(\mathbf{E}_\lambda,\mathbf{H}_\lambda,\boldsymbol{a}_\lambda\big) \in C^1\big([0,T); H^s(\mathbb{R}^3)\times H^s(\mathbb{R}^3)\times(L^2\cap L^\infty)(\Omega_{\text{act}})\big) \qquad \forall s\geq 1. \tag{4.83}$$

Moreover, the triplet $\big(\mathbf{E}_\lambda,\mathbf{H}_\lambda,\boldsymbol{a}_\lambda\big)$ satisfies the relations

$$\forall\, t\in[0,T): \qquad \mathsf{P}_\parallel \mathbf{H}_\lambda(t) = 0, \qquad \mathsf{P}_\parallel\big(\mathbf{E}_\lambda(t) + \mathsf{S}^\lambda \mathsf{G}\boldsymbol{a}_\lambda(t)\big) = 0. \tag{4.84}$$

Proof. (i) We prove that for every $\lambda\geq 1$, the operator $\mathcal{T}^\lambda$ is locally Lipschitz continuous on the space $L^2_\lambda(\mathbb{R}^3)\times L^2_\lambda(\mathbb{R}^3)\times(L^2\cap L^\infty))(\Omega_{\text{act}})$. Recalling $C_g := \max\{C_f,\, L_0,\, L_1\}$ we get from Hypothesis 4.1.5 (iii) and Remark 4.1.6 (i) that for all $\boldsymbol{a}_1,\, \boldsymbol{a}_2 \in \mathbb{R}^3_{\boldsymbol{a}}$ and for all $\mathbf{g}_{\mathbf{E}1},\, \mathbf{g}_{\mathbf{E}2} \in \mathbb{R}^3_{\boldsymbol{a}}$ it holds

$$\big|f(\boldsymbol{a}_1,\mathbf{g}_{\mathbf{E}1}) - f(\boldsymbol{a}_2,\mathbf{g}_{\mathbf{E}2})\big| \leq C_g\big(|\mathbf{g}_{\mathbf{E}1}-\mathbf{g}_{\mathbf{E}2}| + (1+|\mathbf{g}_{\mathbf{E}2}|)\,|\boldsymbol{a}_1-\boldsymbol{a}_2|\big). \tag{4.85}$$

Thus, recalling estimate (4.78) and introducing the constant $C(\lambda) = C_g{\cdot}\max\{1,\, C_{\text{bsl}}\,\lambda^{3/2}\}$, for $(\mathbf{E}_j,\mathbf{H}_j,\boldsymbol{a}_j)\in L^2_\lambda(\mathbb{R}^3)\times L^2_\lambda(\mathbb{R}^3)\times(L^2\cap L^\infty)(\Omega_{\text{act}})$, $j\in\{1,2\}$, we have[10]

$$\begin{aligned}
&\big\|f(\boldsymbol{a}_1,\mathsf{G}^*\mathbf{E}_1) - f(\boldsymbol{a}_2,\mathsf{G}^*\mathbf{E}_2)\big\|_{L^2(\Omega_{\text{act}})} \\
&\qquad\qquad \leq C(\lambda)\Big(C_{\mathsf{G}}\big\|\mathbf{E}_1-\mathbf{E}_2\big\|_{L^2} + (1+C_{\mathsf{G}}\big\|\mathbf{E}_2\big\|_{L^2})\,\|\boldsymbol{a}_1-\boldsymbol{a}_2\|_{L^2(\Omega_{\text{act}})}\Big), \qquad (4.86)\\
&\big\|f(\boldsymbol{a}_1,\mathsf{G}^*\mathbf{E}_1) - f(\boldsymbol{a}_2,\mathsf{G}^*\mathbf{E}_2)\big\|_{L^\infty(\Omega_{\text{act}})} \\
&\qquad\qquad \leq C(\lambda)\Big(C_{\mathsf{G}}\big\|\mathbf{E}_1-\mathbf{E}_2\big\|_{L^2} + (1+C_{\mathsf{G}}\big\|\mathbf{E}_2\big\|_{L^2})\,\|\boldsymbol{a}_1-\boldsymbol{a}_2\|_{L^\infty(\Omega_{\text{act}})}\Big). \qquad (4.87)
\end{aligned}$$

Furthermore, due to the Bernstein type lemma, in $L^2_\lambda(\mathbb{R}^3)$ the differential operator curl is bounded by $C^2_{\text{bsl}}\lambda$.

[10]We recall the convention $\|\cdot\|_{L^p} := \|\cdot\|_{L^p(\mathbb{R}^3)}$ from the beginning of this chapter.

Thus, for $(\mathbf{E}_j, \mathbf{H}_j) \in L^2_\lambda(\mathbb{R}^3) \times L^2_\lambda(\mathbb{R}^3)$, $j \in \{1,2\}$ we have

$$\big\| \operatorname{curl} \mathbf{E}_1 - \operatorname{curl} \mathbf{E}_2 \big\|_{L^2} \le C^2_{\mathrm{bsl}} \lambda \, \big\| \mathbf{E}_1 - \mathbf{E}_2 \big\|_{L^2}, \tag{4.88}$$

$$\big\| \operatorname{curl} \mathbf{H}_1 - \operatorname{curl} \mathbf{H}_2 \big\|_{L^2} \le C^2_{\mathrm{bsl}} \lambda \, \big\| \mathbf{H}_1 - \mathbf{H}_2 \big\|_{L^2}. \tag{4.89}$$

Summarizing, we have that for all $(\mathbf{E}_j, \mathbf{H}_j, \boldsymbol{a}_j) \in L^2_\lambda(\mathbb{R}^3) \times L^2_\lambda(\mathbb{R}^3) \times (L^2 \cap L^\infty)(\Omega_{\mathrm{act}})$, $j \in \{1,2\}$ it holds

$$\begin{aligned} &\big\| \mathcal{T}^\lambda(\mathbf{E}_1, \mathbf{H}_1, \boldsymbol{a}_1) - \mathcal{T}^\lambda(\mathbf{E}_2, \mathbf{H}_2, \boldsymbol{a}_2) \big\|_{L^2 \times L^2 \times (L^2 + L^\infty)(\Omega_{\mathrm{act}})} \\ &\le C^\lambda_L \Big(\big\| \mathbf{H}_1 - \mathbf{H}_2 \big\|_{L^2} + \big\| \mathbf{E}_1 - \mathbf{E}_2 \big\|_{L^2} + \| \boldsymbol{a}_1 - \boldsymbol{a}_2 \|_{L^2(\Omega_{\mathrm{act}})} + \| \boldsymbol{a}_1 - \boldsymbol{a}_2 \|_{L^\infty(\Omega_{\mathrm{act}})} \Big) \end{aligned} \tag{4.90}$$

for a constant $C^\lambda_L = C^\lambda_L(\lambda, C_{\mathsf{G}}, C_g, C_{\mathrm{bsl}}, \|\mathbf{E}_2\|_{L^2})$. This means that for every $\lambda \ge 1$ the operator $\mathcal{T}^\lambda$ is locally[11] Lipschitz continuous on the space $L^2_\lambda(\mathbb{R}^3) \times L^2_\lambda(\mathbb{R}^3) \times (L^2 \cap L^\infty)(\Omega_{\mathrm{act}})$. The Picard-Lindelöf theorem thus yields the existence of a maximal time $T > 0$ and a unique solution

$$\big(\mathbf{E}_\lambda, \mathbf{H}_\lambda, \boldsymbol{a}_\lambda\big) \in C^1\big([0,T); L^2_\lambda(\mathbb{R}^3) \times L^2_\lambda(\mathbb{R}^3) \times (L^2 \cap L^\infty)(\Omega_{\mathrm{act}})\big) \tag{4.91}$$

to the initial value problem (4.81). The Bernstein type lemma yields that for all $\lambda \ge 1$ and for all $s \ge 1$ we have $L^2_\lambda(\mathbb{R}^3) \subset H^s(\mathbb{R}^3)$. Thus, $\big(\mathbf{E}_\lambda, \mathbf{H}_\lambda, \boldsymbol{a}_\lambda\big) \in C^1\big([0,T); H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3) \times (L^2 \cap L^\infty)(\Omega_{\mathrm{act}})\big)$.

(ii) We prove the remaining relations (4.84). The triplet $\big(\mathbf{E}_\lambda, \mathbf{H}_\lambda, \boldsymbol{a}_\lambda\big)$ satisfies the equations (4.73) in a strong sense. Moreover, for all $u \in H^1(\mathbb{R}^3)$ we have $\mathsf{P}_\parallel \operatorname{curl} u = 0$ and obviously the projector $\mathsf{P}_\parallel$ commutes with ∂_t. Due to the inclusion $L^2_\lambda(\mathbb{R}^3) \subset H^1(\mathbb{R}^3)$, this implies

$$\partial_t \mathsf{P}_\parallel \mathbf{E}_\lambda(x,t) + \partial_t \mathsf{P}_\parallel \big(\mathsf{S}^\lambda \mathsf{G} \boldsymbol{a}_\lambda(x,t) \big) = 0, \qquad \partial_t \mathsf{P}_\parallel \mathbf{H}_\lambda(x,t) = 0. \tag{4.92}$$

Since the initial data $\big(\mathbf{E}^\lambda_0, \mathbf{H}^\lambda_0, \boldsymbol{a}^\lambda_0\big)$ satisfies the relation (4.76), we get that (4.84) holds for all $t \in [0,T)$. □

Remark 4.3.2. *Since the triplet* $\big(\mathbf{E}_\lambda, \mathbf{H}_\lambda, \boldsymbol{a}_\lambda\big)$ *satisfies the equations* (4.73a)–(4.73b) *in a strong sense, it is clear that the triplet is also a distributional solution to* (4.73a)–(4.73b). *This means for all* $\psi \in C^\infty_c([0,T) \times \mathbb{R}^3; \mathbb{R}^6)$ *the triplet* $\big(\mathbf{E}_\lambda, \mathbf{H}_\lambda, \boldsymbol{a}_\lambda\big)$ *satisfies*

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3} \partial_t \psi(x,s) \cdot \big(\mathbf{E}_\lambda(x,s), \mathbf{H}_\lambda(x,s)\big) \, dx \, ds + \int_{\mathbb{R}^3} \psi(x,0) \cdot \big(\mathbf{E}^\lambda_0(x), \mathbf{H}^\lambda_0(x)\big) \, dx \\ &- \int_0^T \int_{\mathbb{R}^3} \mathsf{B}^*[\psi(x,s)] \cdot \big(\mathbf{E}_\lambda(x,s), \mathbf{H}_\lambda(x,s)\big) \, dx \, ds \\ &= \int_0^T \int_{\mathbb{R}^3} \psi(x,s) \cdot \Big(\mathsf{S}^\lambda \mathsf{G} f\big(\boldsymbol{a}_\lambda(x,s), \mathbf{E}_\lambda(x,s)\big), 0 \Big) \, dx \, ds. \end{aligned} \tag{4.93}$$

[11] The locality is due to the appearance of $\|\mathbf{E}_2\|_{L^2}$ in the Lipschitz constant C^λ_L.

The next lemma is concerned with uniform (in λ) estimates of the approximating solutions from Lemma 4.3.1.

Lemma 4.3.3. *For all $\lambda \geq 1$ let $\big(\mathbf{E}_\lambda, \mathbf{H}_\lambda, \boldsymbol{a}_\lambda\big)$ denote the unique solution to* (4.81) *and let T denote the corresponding final time. Then, the following holds*

(i) For all $t \in [0, T)$ and for all $\lambda \geq 1$ we have $\|\boldsymbol{a}_\lambda(t)\|_{L^\infty(\Omega_{\mathrm{act}})} \leq C_b$ with C_b from Hypothesis 4.1.5.

(ii) For all $t \in [0, T)$ and for all $\lambda \geq 1$ the following estimate holds for a constant $C_{\mathrm{uni}} = C_{\mathrm{uni}}(T, C_g, C_b, C_{\mathsf{G}}, \Omega_{\mathrm{act}})$

$$
\begin{aligned}
&\big\|\big(\mathbf{E}_\lambda(t), \mathbf{H}_\lambda(t)\big)\big\|_{L^2}^2 + \big\|\boldsymbol{a}(t)\big\|_{L^2(\Omega_{\mathrm{act}})}^2 \\
&\qquad\qquad \leq C_{\mathrm{uni}}\Big(1 + \big\|\big(\mathbf{E}_0^\lambda, \mathbf{H}_0^\lambda\big)\big\|_{L^2(\mathbb{R}^3)}^2 + \big\|\boldsymbol{a}_0^\lambda\big\|_{L^2(\Omega_{\mathrm{act}})}^2\Big) \qquad (4.94)\\
&\qquad\qquad \leq C_{\mathrm{uni}}\Big(1 + \big\|\big(\mathbf{E}_0, \mathbf{H}_0\big)\big\|_{L^2(\mathbb{R}^3)}^2 + \big\|\boldsymbol{a}_0\big\|_{L^2(\Omega_{\mathrm{act}})}^2\Big). \qquad (4.95)
\end{aligned}
$$

Proof. (i) Can be proved exactly as Proposition 4.1.10 (iii).

(ii) The second estimate (4.95) is clear due to (4.73d) and Lemma A.3.13 (iii). To prove the first estimate, note that the operator B is symmetric, i.e. we have $\mathsf{B} = \sum_{j=1}^3 A_j \partial_{x_j}$ with the symmetric matrices from (A.144). For the smooth solutions $\big(\mathbf{E}_\lambda, \mathbf{H}_\lambda\big)$ from Lemma 4.3.1, we thus get that for all $t \in [0, T)$ and for all $\lambda \geq 1$ it holds[12]

$$
\big\|\big(\mathbf{E}_\lambda(t), \mathbf{H}_\lambda(t)\big)\big\|_{L^2}^2 = \big\|\big(\mathbf{E}_0^\lambda, \mathbf{H}_0^\lambda\big)\big\|_{L^2}^2 - 2\int_0^t \int_{\mathbb{R}^3} \mathsf{S}^\lambda \mathsf{G} f(\boldsymbol{a}_\lambda, \mathsf{G}^* \mathbf{E}_\lambda) \cdot \mathbf{E}_\lambda \, dx \, ds. \qquad (4.96)
$$

Furthermore, from Remark 4.1.6 (ii) and the first result of the present lemma we get the following estimate with a constant $C_{L^2} = C_{L^2}(C_g, C_b, \Omega_{\mathrm{act}})$ by involving Young's inequality

$$
\begin{aligned}
\forall\, t \in [0, T),\ \forall\, \lambda \geq 1: \quad &\big\|f(\boldsymbol{a}_\lambda(t), \mathsf{G}^*\mathbf{E}_\lambda(t))\big\|_{L^2(\Omega_{\mathrm{act}})}^2 \\
&\leq C_g^2 \Big\|\big(1 + |\boldsymbol{a}_\lambda(t)|\big)\big(1 + |\mathsf{G}^*\mathbf{E}_\lambda(t)|\big)\Big\|_{L^2(\Omega_{\mathrm{act}})}^2 \\
&\leq C_{L^2}\Big(1 + \big\|\boldsymbol{a}_\lambda(t)\big\|_{L^2(\Omega_{\mathrm{act}})}^2 + \big\|\mathsf{G}^*\mathbf{E}_\lambda(t)\big\|_{L^2(\Omega_{\mathrm{act}})}^2\Big). \qquad (4.97)
\end{aligned}
$$

[12] For more details of the exact calculations we refer to the proof of Proposition A.5.3.

The above estimate implies the following two estimates for all $t \in [0,T)$ and for all $\lambda \geq 1$

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^3} \big|\mathsf{S}^\lambda \mathsf{G} f(\boldsymbol{a}_\lambda, \mathsf{G}^* \mathbf{E}_\lambda)\big| \, \big|\mathbf{E}_\lambda\big| \, dx \, ds \\
&\leq \frac{1}{2} \int_0^t \big\|\mathsf{S}^\lambda \mathsf{G} f(\boldsymbol{a}_\lambda(s), \mathsf{G}^* \mathbf{E}_\lambda(s))\big\|_{L^2}^2 + \big\|\mathbf{E}_\lambda(s)\big\|_{L^2}^2 \, ds \\
&\leq \int_0^t C_{\mathsf{G}} \big\|f(\boldsymbol{a}_\lambda(s), \mathsf{G}^* \mathbf{E}_\lambda(s))\big\|_{L^2(\Omega_{\mathrm{act}})}^2 + \big\|\mathbf{E}_\lambda(s)\big\|_{L^2}^2 \, ds \\
&\leq C(C_b, C_{\mathsf{G}}, C_g, \Omega_{\mathrm{act}}) \int_0^t \Big(1 + \big\|\boldsymbol{a}_\lambda(s)\big\|_{L^2(\Omega_{\mathrm{act}})}^2 + \big\|\mathsf{G}^* \mathbf{E}_\lambda(s)\big\|_{L^2(\Omega_{\mathrm{act}})}^2 + \big\|\mathbf{E}_\lambda(s)\big\|_{L^2}^2\Big) \, ds \\
&\leq C(C_b, C_{\mathsf{G}}, C_g, \Omega_{\mathrm{act}}) \int_0^t \Big(1 + \big\|\boldsymbol{a}_\lambda(s)\big\|_{L^2(\Omega_{\mathrm{act}})}^2 + \big\|\mathbf{E}_\lambda(s)\big\|_{L^2}^2\Big) \, ds
\end{aligned} \tag{4.98}
$$

and

$$
\begin{aligned}
&\int_0^t \int_{\Omega_{\mathrm{act}}} \big|f(\boldsymbol{a}_\lambda, \mathsf{G}^* \mathbf{E}_\lambda)\big| \, \big|\boldsymbol{a}_\lambda\big| \, dx \, ds \\
&\leq \frac{1}{2} \int_0^t \big\|f(\boldsymbol{a}_\lambda(s), \mathsf{G}^* \mathbf{E}_\lambda(s))\big\|_{L^2(\Omega_{\mathrm{act}})}^2 + \big\|\boldsymbol{a}_\lambda(s)\big\|_{L^2(\Omega_{\mathrm{act}})}^2 \, ds \\
&\leq C(C_b, C_{\mathsf{G}}, C_g, \Omega_{\mathrm{act}}) \int_0^t \Big(1 + \big\|\boldsymbol{a}_\lambda(s)\big\|_{L^2(\Omega_{\mathrm{act}})}^2 + \big\|\mathbf{E}_\lambda(s)\big\|_{L^2}^2\Big) \, ds.
\end{aligned} \tag{4.99}
$$

For all $t \in (0,T)$ we define $Q_t := (0,t) \times \mathbb{R}^3$ and $\Omega_t := (0,t) \times \Omega_{\mathrm{act}}$. Multiplying equation (4.73c) with $\boldsymbol{a}_\lambda$, integrating over Ω_t and adding the resulting equation to (4.96), yields the following estimate for all $t \in [0,T)$ and for all $\lambda \geq 1$ with an obvious constant $C > 0$

$$
\begin{aligned}
&\big\|\big(\mathbf{E}_\lambda(t), \mathbf{H}_\lambda(t)\big)\big\|_{L^2}^2 + \big\|\boldsymbol{a}(t)\big\|_{L^2(\Omega_{\mathrm{act}})}^2 \\
&\leq \Big(\big\|\big(\mathbf{E}_0^\lambda, \mathbf{H}_0^\lambda\big)\big\|_{L^2(\mathbb{R}^3)}^2 + \big\|\boldsymbol{a}_0^\lambda\big\|_{L^2(\Omega_{\mathrm{act}})}^2\Big) + C \int_0^t \Big(1 + \big\|\boldsymbol{a}_\lambda(s)\big\|_{L^2(\Omega_{\mathrm{act}})}^2 + \big\|\mathbf{E}_\lambda(s)\big\|_{L^2}^2\Big) \, ds \\
&\leq \Big(\big\|\big(\mathbf{E}_0^\lambda, \mathbf{H}_0^\lambda\big)\big\|_{L^2(\mathbb{R}^3)}^2 + \big\|\boldsymbol{a}_0^\lambda\big\|_{L^2(\Omega_{\mathrm{act}})}^2 + CT\Big) \\
&\qquad + C \int_0^t \Big(\big\|\boldsymbol{a}_\lambda(s)\big\|_{L^2(\Omega_{\mathrm{act}})}^2 + \big\|\mathbf{E}_\lambda(s)\big\|_{L^2}^2 + \big\|\mathbf{H}_\lambda(s)\big\|_{L^2}^2\Big) \, ds.
\end{aligned} \tag{4.100}
$$

Gronwall's lemma (see Theorem A.1.9) yields the assertion. $\square$

The above lemma implies global (in time) existence for $\big(\mathbf{E}_\lambda, \mathbf{H}_\lambda, \boldsymbol{a}_\lambda\big)$, i.e. $T = \infty$.

4.3.2. Preparation and Weak Convergence

We fix an arbitrary $T > 0$ for the rest of this proof and set $Q_T := (0,T) \times \mathbb{R}^3$ and $\Omega_T := (0,T) \times \Omega_{\mathrm{act}}$. From Lemma 4.3.1 we get that for every $\lambda \geq 1$ there exists a unique solution $\big(\mathbf{E}_\lambda, \mathbf{H}_\lambda, \boldsymbol{a}_\lambda\big)$ to the approximating Cauchy problem (4.73). Thus, $\{(\mathbf{E}_\lambda, \mathbf{H}_\lambda, \boldsymbol{a}_\lambda)\}_{\lambda \geq 1}$ defines a family of solutions to approximating problems.

In this subsection we construct a subsequence of the family $\{(\mathbf{E}_\lambda, \mathbf{H}_\lambda, \boldsymbol{a}_\lambda)\}_{\lambda \geq 1}$ that weakly converges to a triplet $\big(\mathbf{E}_\infty, \mathbf{H}_\infty, \boldsymbol{a}_\infty\big)$ in a suitable sense. In the following subsections we will show that the convergence is in fact strong (Sec. 4.3.3–4.3.5) and that the triplet $\big(\mathbf{E}_\infty, \mathbf{H}_\infty, \boldsymbol{a}_\infty\big)$ is a solution to Problem 4.1.8 in the sense of Definition 4.1.9 (Sec. 4.3.6).

Estimate (4.95) from Lemma 4.3.3 yields the existence of a subsequence (still denoted with λ) $\{(\mathbf{E}_\lambda, \mathbf{H}_\lambda)\}_{\lambda \geq 1} \subset L^2(Q_T)$ and a pair $\big(\mathbf{E}_\infty, \mathbf{H}_\infty\big) \in L^2(Q_T)$ such that

$$\big(\mathbf{E}_\lambda, \mathbf{H}_\lambda\big) \longrightarrow \big(\mathbf{E}_\infty, \mathbf{H}_\infty\big) \quad \text{weakly in } L^2(Q_T). \tag{4.101}$$

Moreover, estimate (4.95) and Lemma 4.3.3 (i) yield the existence of a subsequence of $\{\boldsymbol{a}_\lambda\}_{\lambda \geq 1} \subset (L^2 \cap L^\infty)(\Omega_{\mathrm{act}})$ (still denoted with λ) and some $\boldsymbol{a}_\infty \in (L^2 \cap L^\infty)(\Omega_{\mathrm{act}})$ such that

$$\boldsymbol{a}_\lambda \longrightarrow \boldsymbol{a}_\infty \qquad \begin{cases} \text{weakly in } L^2(\Omega_T) \quad \text{and} \\ \text{weakly-}* \text{ in } L^\infty(\Omega_T). \end{cases} \tag{4.102}$$

Moreover, due to the strong convergence $\forall\, \mathbf{F} \in L^2 : \ \big\|\mathsf{S}^\lambda \mathbf{F} - \mathbf{F}\big\|_{L^2} \longrightarrow 0$ (see Lemma A.3.13 (iv)) it holds

$$\big(\mathbf{E}_0^\lambda, \mathbf{H}_0^\lambda, \boldsymbol{a}_0^\lambda\big) \longrightarrow \big(\mathbf{E}_0, \mathbf{H}_0, \boldsymbol{a}_0\big) \quad \text{strongly in } L^2 \times L^2 \times L^2(\Omega_{\mathrm{act}}). \tag{4.103}$$

Declaration 4.3.4. *Let* $\big\{\big(\mathbf{E}_\lambda, \mathbf{H}_\lambda, \boldsymbol{a}_\lambda\big)\big\}_{\lambda \geq 1} \subset L^2(Q_T) \times L^2(Q_T) \times (L^2 \cap L^\infty)(\Omega_T)$ *denote a subsequence of unique solutions to the approximating Cauchy problems* (4.73) *that satisfies the convergences*

$$\big(\mathbf{E}_\lambda, \mathbf{H}_\lambda\big) \longrightarrow \big(\mathbf{E}_\infty, \mathbf{H}_\infty\big) \qquad \textit{weakly in } L^2(Q_T) \times L^2(Q_T), \tag{4.104a}$$

$$\boldsymbol{a}_\lambda \longrightarrow \boldsymbol{a}_\infty \qquad \begin{cases} \textit{weakly in } L^2(\Omega_T) \quad \textit{and} \\ \textit{weakly-}* \textit{ in } L^\infty(\Omega_T). \end{cases} \tag{4.104b}$$

Moreover, we introduce the following constants

$$C_{\boldsymbol{a}} := \sup_{\lambda \geq 1} \big\|\boldsymbol{a}_\lambda\big\|_{L^2(\Omega_T)}, \qquad C_{\mathbf{E}} := \sup_{\lambda \geq 1} \big\|\mathbf{E}_\lambda\big\|_{L^2(Q_T)}, \tag{4.105}$$

and recall that for all $\lambda \geq 1$ we have

$$\big(\mathbf{E}_\lambda, \mathbf{H}_\lambda, \boldsymbol{a}_\lambda\big) \in C^1([0,T]; H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3) \times (L^2 \cap L^\infty)(\Omega_{\mathrm{act}})) \qquad \forall s \geq 1. \tag{4.106}$$

In the next subsection we will prove that the subsequence $\{\boldsymbol{a}_\lambda\}_{\lambda\geq 1}$ from Declaration 4.3.4 strongly converges to $\boldsymbol{a}_\infty$ in $L^2(\Omega_T)$. As a preparation for this proof, we introduce a weight, and a corresponding weighted norm. The introduction of this weighted norm is a key idea in the whole existence proof.[13] First, we motivate this approach.

In the following let $\boldsymbol{a}_\lambda$ and $\boldsymbol{a}_\mu$ denote the unique solutions to (4.73) with S^λ and S^μ, respectively.[14] Then, from the Lipschitz estimates of Hypothesis 4.1.5 and Remark 4.1.6 we get the following pointwise estimate for $\boldsymbol{a}_j$, $\mathsf{G}^*\mathbf{E}_j \in \mathbb{R}^3_{\boldsymbol{a}}$ with $j \in \{\lambda, \mu\}$

$$\begin{aligned} f(\boldsymbol{a}_\lambda, \mathsf{G}^*\mathbf{E}_\lambda) - f(\boldsymbol{a}_\mu, \mathsf{G}^*\mathbf{E}_\mu) &= \big(f_0(\boldsymbol{a}_\lambda) + f_1(\boldsymbol{a}_\lambda)\,\mathsf{G}^*\mathbf{E}_\lambda\big) - \big(f_0(\boldsymbol{a}_\mu) + f_1(\boldsymbol{a}_\mu)\,\mathsf{G}^*\mathbf{E}_\mu\big) \\ &= \big(f_0(\boldsymbol{a}_\lambda) - f_0(\boldsymbol{a}_\mu)\big) + \big(f_1(\boldsymbol{a}_\lambda) - f_1(\boldsymbol{a}_\mu)\big)\,\mathsf{G}^*\mathbf{E}_\infty \\ &\quad + f_1(\boldsymbol{a}_\lambda)\big(\mathsf{G}^*\mathbf{E}_\lambda - \mathsf{G}^*\mathbf{E}_\infty\big) - f_1(\boldsymbol{a}_\mu)\big(\mathsf{G}^*\mathbf{E}_\mu - \mathsf{G}^*\mathbf{E}_\infty\big) \\ &\leq L_0\big|\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big| + L_1\big|\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big|\,\big|\mathsf{G}^*\mathbf{E}_\infty\big| \\ &\quad + f_1(\boldsymbol{a}_\lambda)\big(\mathsf{G}^*\mathbf{E}_\lambda - \mathsf{G}^*\mathbf{E}_\infty\big) - f_1(\boldsymbol{a}_\mu)\big(\mathsf{G}^*\mathbf{E}_\mu - \mathsf{G}^*\mathbf{E}_\infty\big) \\ &\leq L\big(1 + |\mathsf{G}^*\mathbf{E}_\infty|\big)\big|\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big| \\ &\quad + f_1(\boldsymbol{a}_\lambda)\big(\mathsf{G}^*\mathbf{E}_\lambda - \mathsf{G}^*\mathbf{E}_\infty\big) - f_1(\boldsymbol{a}_\mu)\big(\mathsf{G}^*\mathbf{E}_\mu - \mathsf{G}^*\mathbf{E}_\infty\big) \end{aligned} \tag{4.107}$$

with $L := \max\{L_0, L_1\}$. Plugging this pointwise estimate into the difference of the equations for $\boldsymbol{a}_\lambda$ and $\boldsymbol{a}_\mu$ in (4.70c) yields the pointwise estimate

$$\begin{aligned} \frac{1}{2}\partial_t\big|\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big|^2 &= \big(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big)\cdot\big(f(\boldsymbol{a}_\lambda, \mathsf{G}^*\mathbf{E}_\lambda) - f(\boldsymbol{a}_\mu, \mathsf{G}^*\mathbf{E}_\mu)\big) \\ &\leq L\big(1 + |\mathsf{G}^*\mathbf{E}_\infty|\big)\big|\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big|^2 \\ &\quad + \Big(f_1(\boldsymbol{a}_\lambda)\big(\mathsf{G}^*\mathbf{E}_\lambda - \mathsf{G}^*\mathbf{E}_\infty\big) - f_1(\boldsymbol{a}_\mu)\big(\mathsf{G}^*\mathbf{E}_\mu - \mathsf{G}^*\mathbf{E}_\infty\big)\Big)\cdot\big(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big). \end{aligned} \tag{4.108}$$

The quadratic term $|\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu|^2$ can be absorbed by introducing a weight $e^{-b(x,t)}$ with a function b depending on $(x,t) \in Q_T$ that satisfies $\partial_t b(x,t) = L\big(1 + |\mathsf{G}^*\mathbf{E}_\infty(x,s)|\big)$ in Ω_T. We choose the scalar, positive, measurable and almost everywhere finite function

$$b(x,t) := \begin{cases} L\int_0^t \big(1 + |\mathsf{G}^*\mathbf{E}_\infty(x,s)|\big)\,ds, & (x,t) \in \Omega_T \\ \infty, & (x,t) \in Q_T \setminus \Omega_T. \end{cases} \tag{4.109}$$

This precise choice serves two purposes. Firstly, the weight e^{-b} absorbs the quadratic term $|\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu|^2$ in equation (4.108) in the set Ω_T and secondly the weight e^{-b} is such that[15]

$$\forall\, t \in [0,T]: \qquad x \longmapsto e^{-b(x,t)} \in L^p(\mathbb{R}^3) \qquad \forall\, p \in [1,\infty]. \tag{4.110}$$

[13] See also [Joc02b], [JMR00a], [Dum05] and [DuS12].
[14] We slightly abuse our notation in the next pointwise estimates by writing $\mathsf{G}^*\mathbf{E}$ for an element of $\mathbb{R}^3_{\boldsymbol{a}}$.
[15] This will be important in the estimates (4.147) and (4.165). Namely, we can approximate the function $(x,t) \longmapsto e^{-b(x,t)}$ with a function from the space $C_c^\infty(\Omega_T)$ in $L^2(\Omega_T)$ and with a function from the space $C_c^\infty(Q_T)$ in $L^4(Q_T)$.

By including the function e^{-b}, we get the following pointwise estimate from (4.108)

$$\begin{aligned}\frac{1}{2}\partial_t\big(e^{-2b}|\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu|^2\big) &= e^{-2b}\frac{1}{2}(-2\partial_t b)|\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu|^2 + e^{-2b}\frac{1}{2}\partial_t|\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu|^2 \\ &= -e^{-2b}L\big(1 + |\mathsf{G}^*\mathbf{E}_\infty|\big)\big|\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big|^2 + e^{-2b}\frac{1}{2}\partial_t\big|\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big|^2 \\ &\le e^{-2b}\Big(f_1(\boldsymbol{a}_\lambda)\big(\mathsf{G}^*\mathbf{E}_\lambda - \mathsf{G}^*\mathbf{E}_\infty\big) - f_1(\boldsymbol{a}_\mu)\big(\mathsf{G}^*\mathbf{E}_\mu - \mathsf{G}^*\mathbf{E}_\infty\big)\Big)\cdot\big(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big). \quad (4.111)\end{aligned}$$

Integrating this estimate over $\Omega_t := (0,t)\times\Omega_{\mathrm{act}}$ for some arbitrary $t \in (0,T)$ yields the estimate[16]

$$\begin{aligned}&\frac{1}{2}\Big\|e^{b(t)}\big(\boldsymbol{a}_\lambda(t) - \boldsymbol{a}_\mu(t)\big)\Big\|^2_{L^2(\Omega_{\mathrm{act}})} \\ &\le \int_{\Omega_t} e^{-2b}\Big(f_1(\boldsymbol{a}_\lambda)\big(\mathsf{G}^*\mathbf{E}_\lambda - \mathsf{G}^*\mathbf{E}_\infty\big) - f_1(\boldsymbol{a}_\mu)\big(\mathsf{G}^*\mathbf{E}_\mu - \mathsf{G}^*\mathbf{E}_\infty\big)\Big)\cdot\big(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big)\,dx\,ds \\ &\quad= \int_{\Omega_t} e^{-2b}f_1(\boldsymbol{a}_\lambda)\big(\mathsf{G}^*\mathbf{E}_\lambda - \mathsf{G}^*\mathbf{E}_\infty\big)\cdot\big(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big)\,dx\,ds \qquad (4.112\text{a}) \\ &\qquad - \int_{\Omega_t} e^{-2b}f_1(\boldsymbol{a}_\mu)\big(\mathsf{G}^*\mathbf{E}_\mu - \mathsf{G}^*\mathbf{E}_\infty\big)\cdot\big(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big)\,dx\,ds, \qquad (4.112\text{b})\end{aligned}$$

since we have $\boldsymbol{a}_0^\lambda = \boldsymbol{a}_0^\mu$ in view of (4.73d).

4.3.3. Strong Convergence of the Bloch Vector

In this section we prove the following convergence result.

Proposition 4.3.5. *The sequence $\{\boldsymbol{a}_\lambda\}_{\lambda\ge 1}$ from Declaration 4.3.4 satisfies*

$$\big\|\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big\|_{L^2(\Omega_T)} \longrightarrow 0 \qquad \text{as } \lambda,\mu\to\infty. \qquad (4.113)$$

The proof is based on the following two observations.

Lemma 4.3.6. *The equivalence class $\boldsymbol{a}_\infty \in L^2((0,T);L^2(\Omega_{\mathrm{act}}))$ from Declaration 4.3.4 has a continuous representative $\widetilde{\boldsymbol{a}}_\infty \in C^0([0,T];L^2(\Omega_{\mathrm{act}}))$ and we have the convergence*

$$\forall\, t\in[0,T]: \qquad \boldsymbol{a}_\lambda(t) \longrightarrow \widetilde{\boldsymbol{a}}_\infty(t) \qquad \text{weakly in } L^2(\Omega_{\mathrm{act}}). \qquad (4.114)$$

Proof. In view of Corollary A.1.7 we need to show that $\{\boldsymbol{a}_\lambda\}_{\lambda\ge1} \subset W^{1,\infty}([0,T];L^2(\Omega_{\mathrm{act}}))$ is bounded. For all $\lambda \ge 1$ we have $\boldsymbol{a}_\lambda \in C^1([0,T];L^2(\Omega_{\mathrm{act}}))$ and due to estimate (4.95) from Lemma 4.3.3 we have

$$\sup_{\lambda\ge1}\|\boldsymbol{a}_\lambda\|_{C^0([0,T];L^2(\Omega_{\mathrm{act}}))} \le C_{\mathrm{uni}}^{1/2}\Big(1+\big\|\big(\mathbf{E}_0,\mathbf{H}_0\big)\big\|^2_{L^2(\mathbb{R}^3)}+\big\|\boldsymbol{a}_0\big\|^2_{L^2(\Omega_{\mathrm{act}})}\Big)^{1/2} =: C_{\boldsymbol{a}}^{\mathrm{uni}}. \quad (4.115)$$

[16] From now on, we get back to our regular notation.

We recall that the constant C_{uni} does not depend on λ (see Lemma 4.3.3). Moreover, from the L^2-estimate (4.97) and the uniform in time estimate (4.95) we get the following estimate with the constant $C_{L^2} = C_{L^2}(C_g, C_b, \Omega_{\mathrm{act}})$ from (4.97)

$$\begin{aligned}\forall\, t \in [0,T],\ \forall \lambda \geq 1: \quad \|\partial_t \boldsymbol{a}_\lambda(t)\|_{L^2(\Omega_{\mathrm{act}})} &= \big\| f(\boldsymbol{a}_\lambda(t), \mathsf{G}^* \mathbf{E}_\lambda(t))\big\|_{L^2(\Omega_{\mathrm{act}})} \\ &\leq \sqrt{C_{L^2}} \Big(1 + \big\|\boldsymbol{a}(t)\big\|^2_{L^2(\Omega_{\mathrm{act}})} + \big\|\mathsf{G}^*\mathbf{E}(t)\big\|^2_{L^2(\Omega_{\mathrm{act}})}\Big)^{1/2} \\ &\leq \sqrt{C_{L^2}} \Big(1 + \big\|\boldsymbol{a}(t)\big\|^2_{L^2(\Omega_{\mathrm{act}})} + C_{\mathsf{G}}\big\|\mathbf{E}(t)\big\|^2_{L^2} + \big\|\mathbf{H}(t)\big\|^2_{L^2}\Big)^{1/2} \\ &\leq C\Big(1 + \big\|\boldsymbol{a}_0\big\|^2_{L^2(\Omega_{\mathrm{act}})} + \big\|\big(\mathbf{E}_0, \mathbf{H}_0\big)\big\|^2_{L^2}\Big)^{1/2}. \end{aligned} \tag{4.116}$$

The final constant C depends on $\big\{C_g, C_b, C_{\mathsf{G}}, \Omega_{\mathrm{act}}, C_{\mathrm{uni}}\big\}$ but is independent of λ. This means, the family $\{\boldsymbol{a}_\lambda\}_{\lambda\geq 1}$ is bounded in the space $W^{1,\infty}([0,T]; L^2(\Omega_{\mathrm{act}}))$. From Corollary A.1.7 we may therefore infer the existence of some subsequence $\{\boldsymbol{a}_{\lambda_k}\}_{k\in\mathbb{N}}$ and some function $\widetilde{\boldsymbol{a}}_\infty \in W^{1,\infty}([0,T]; L^2(\Omega_{\mathrm{act}}))$ such that

$$\forall\, t \in [0,T]: \qquad \boldsymbol{a}_{\lambda_k}(t) \longrightarrow \widetilde{\boldsymbol{a}}_\infty(t) \qquad \text{weakly in } L^2(\Omega_{\mathrm{act}}). \tag{4.114}$$

Due to the uniqueness of the limit, we get that $\widetilde{\boldsymbol{a}}_\infty$ is the continuous representative of the equivalence class $\boldsymbol{a}_\infty$, and the convergence (4.114) holds for the full sequence. □

Lemma 4.3.7. *For every $T > 0$ and for every $\delta > 0$ there exists a constant $C_{\mathrm{cru}} = C_{\mathrm{cru}}(T, C_f, C_{\mathsf{G}})$ and an index $N = N(\delta, T) \geq 1$ such that for all $\lambda, \mu \geq N$ and for all $t \in [0,T]$ the following estimate holds*

$$\big\|e^{-b(t)}\big(\boldsymbol{a}_\lambda(t) - \boldsymbol{a}_\mu(t)\big)\big\|^2_{L^2(\Omega_{\mathrm{act}})} \leq C_{\mathrm{cru}} \cdot \Big(\delta + \int_0^t \big\|e^{-b(s)}\big(\boldsymbol{a}_\lambda(s) - \boldsymbol{a}_\infty(s)\big)\big\|^2_{L^2(\Omega_{\mathrm{act}})}\, ds\Big). \tag{4.117}$$

This estimate is the crucial tool in the proof of Proposition 4.3.5. All technical difficulties and key ideas enter in its proof which is the issue of Section 4.3.4. We stress that in the proof of Lemma 4.3.6 we did not use the result from Lemma 4.3.7 and continue proving Proposition 4.3.5.

Proof of Proposition 4.3.5. We identify the equivalence class $\boldsymbol{a}_\infty$ with its continuous representative $\widetilde{\boldsymbol{a}}_\infty$ and neglect the $\widetilde{\cdot}$. Due to Lemma 4.3.6 the convergence $\boldsymbol{a}_\mu(t) \longrightarrow \boldsymbol{a}_\infty(t)$ weakly in $L^2(\Omega_{\mathrm{act}})$ is uniformly in t. The weak lower semi-continuity of the norm thus yields that for all $\lambda \geq 1$ we have

$$\big\|e^{-b(t)}\big(\boldsymbol{a}_\lambda(t) - \boldsymbol{a}_\infty(t)\big)\big\|^2_{L^2(\Omega_{\mathrm{act}})} \leq \liminf_{\mu\to\infty} \big\|e^{-b(t)}\big(\boldsymbol{a}_\lambda(t) - \boldsymbol{a}_\mu(t)\big)\big\|^2_{L^2(\Omega_{\mathrm{act}})}. \tag{4.118}$$

Next, we fix an arbitrary small $\epsilon > 0$ and choose $\delta := \frac{\epsilon}{C_{\mathrm{cru}} \exp(C_{\mathrm{cru}} \cdot T)}$ with C_{cru} from Lemma 4.3.7.

Then, estimate (4.118) and Lemma 4.3.7 yield that for every $\lambda \geq N(\delta, T)$ we have

$$\left\| e^{-b(t)} \big(\boldsymbol{a}_\lambda(t) - \boldsymbol{a}_\infty(t) \big) \right\|^2_{L^2(\Omega_{\text{act}})} \leq C_{\text{cru}} \cdot \Big(\delta + \int_0^t \left\| e^{-b(s)} \big(\boldsymbol{a}_\lambda(s) - \boldsymbol{a}_\infty(s) \big) \right\|^2_{L^2(\Omega_{\text{act}})} ds \Big). \tag{4.119}$$

Using Gronwall's lemma (see Theorem A.1.9), we get that for all $\lambda \geq N$ and for all $t \in [0, T]$ the following estimate holds

$$\left\| e^{-b(t)} \big(\boldsymbol{a}_\lambda(t) - \boldsymbol{a}_\infty(t) \big) \right\|^2_{L^2(\Omega_{\text{act}})} \leq \delta\, C_{\text{cru}} \exp(C_{\text{cru}} \cdot t) \leq \epsilon. \tag{4.120}$$

Thus, we have shown the convergence

$$\left\| e^{-b(t)} \big(\boldsymbol{a}_\lambda(t) - \boldsymbol{a}_\infty(t) \big) \right\|^2_{L^2(\Omega_{\text{act}})} \longrightarrow 0 \qquad \text{uniformly in } t \qquad \text{as } \lambda \to \infty. \tag{4.121}$$

In particular, this implies $\boldsymbol{a}_\lambda \longrightarrow \boldsymbol{a}_\infty$ strongly in $L^2(\Omega_T, e^{-b} dx\, dt)$.

We end this proof by showing the strong convergence $\boldsymbol{a}_\lambda \longrightarrow \boldsymbol{a}_\infty$ in $L^2(\Omega_T)$. To this end, we proceed exactly as in the proof of Lemma 3.4.5 and define

$$\mathbf{A}_\lambda := \int_{\Omega_T} \left| \boldsymbol{a}_\lambda - \boldsymbol{a}_\infty \right|^2 dx\, dt, \qquad \widetilde{\boldsymbol{a}}_\lambda(x,t) := \left| \boldsymbol{a}_\lambda(x,t) - \boldsymbol{a}_\infty(x,t) \right|^2. \tag{4.122}$$

Due to the L^2-bounds on $\{\boldsymbol{a}_\lambda\}_{\lambda \geq 1}$ from (4.105) we see that $\{\mathbf{A}_\lambda\}_{\lambda \geq 1} \subset \mathbb{R}$ is bounded. Hence, there exist a subsequence $\{\mathbf{A}_{\lambda_k}\}_{k \in \mathbb{N}} \subset \{\mathbf{A}_\lambda\}_{\lambda \geq 1}$ and some $\mathbf{A}_\infty \geq 0$ with

$$\mathbf{A}_\infty := \limsup_{\lambda \to \infty} \mathbf{A}_\lambda = \lim_{k \to \infty} \mathbf{A}_{\lambda_k}. \tag{4.123}$$

On the other hand, since b is positive and finite almost everywhere in Ω_T, it holds $e^{-b} \neq 0$ a.e. in Ω_T. Thus, due to the convergence from (4.121) and due to Weyl's theorem, we may extract a subsequence $\{\boldsymbol{a}_{\lambda_k}\}_{k \in \mathbb{N}} \subset \{\boldsymbol{a}_\lambda\}_{\lambda \geq 1}$ that converges pointwise a.e. in Ω_T to $\boldsymbol{a}_\infty$ w.r.t. the measures $dx\, dt$ as well as $e^{-b}\, dx\, dt$. In particular, this implies that we may also extract a sub-subsequence $\{\widetilde{\boldsymbol{a}}_{\lambda_{k_j}}\}_{j \in \mathbb{N}} \subset \{\widetilde{\boldsymbol{a}}_{\lambda_k}\}_{k \in \mathbb{N}}$ that converges pointwise a.e. in Ω_T to zero as $j \to \infty$. Thanks to the L^∞-bounds on $\boldsymbol{a}_\lambda$ from Lemma 4.3.3, we may infer with Lebesgue's convergence theorem the strong convergence $\widetilde{\boldsymbol{a}}_{\lambda_{k_j}} \longrightarrow 0$ in $L^1(\Omega_T)$. Due to the uniqueness of the limit, this implies $\mathbf{A}_\infty = 0$ as well as the convergence

$$\boldsymbol{a}_\lambda \longrightarrow \boldsymbol{a}_\infty \qquad \text{strongly in } L^2(\Omega_T) \qquad \text{as } \lambda \to \infty \tag{4.124}$$

for the full sequence. This proves Proposition 4.3.5. □

4.3.4. Proof of Lemma 4.3.7

We start from estimate (4.112) and show that the sequence $\{\boldsymbol{a}_\lambda\}_{\lambda \geq 1}$ from Declaration 4.3.4 satisfies the estimate (4.117) from Lemma 4.3.7. Key ingredients of the following proof are a result from compensated compactness due to Gérard, a Rellich-type Lemma and the Helmholtz decomposition of the electric field. We recall that the dipole moment operator G^* is linear. Therefore, estimate (4.112) from Section 4.3.2 is equivalent to

$$\frac{1}{2}\Big\|e^{b(t)}\big(\boldsymbol{a}_\lambda(t)-\boldsymbol{a}_\mu(t)\big)\Big\|^2_{L^2(\Omega_{\mathrm{act}})}$$
$$\le \int_{\Omega_t} e^{-2b}\Big(f_1(\boldsymbol{a}_\lambda)\,\mathsf{G}^*\big(\mathbf{E}_\lambda-\mathbf{E}_\infty\big)-f_1(\boldsymbol{a}_\mu)\,\mathsf{G}^*\big(\mathbf{E}_\mu-\mathbf{E}_\infty\big)\Big)\cdot\big(\boldsymbol{a}_\lambda-\boldsymbol{a}_\mu\big)\,dx\,ds$$
$$=\int_{\Omega_t} e^{-2b}f_1(\boldsymbol{a}_\lambda)\,\mathsf{G}^*\big(\mathbf{E}_\lambda-\mathbf{E}_\infty\big)\cdot\big(\boldsymbol{a}_\lambda-\boldsymbol{a}_\mu\big)\,dx\,ds \tag{4.125a}$$
$$-\int_{\Omega_t} e^{-2b}f_1(\boldsymbol{a}_\mu)\,\mathsf{G}^*\big(\mathbf{E}_\mu-\mathbf{E}_\infty\big)\cdot\big(\boldsymbol{a}_\lambda-\boldsymbol{a}_\mu\big)\,dx\,ds. \tag{4.125b}$$

Next, we take a closer look at the terms $f_1(\boldsymbol{a}_\nu)\,\mathsf{G}^*\big(\mathbf{E}_\nu-\mathbf{E}_\infty\big)$ where ν is a placeholder for λ or μ. We split the factor $e^{-2b}f_1(\boldsymbol{a}_\nu)\,\mathsf{G}^*\big(\mathbf{E}_\nu-\mathbf{E}_\infty\big)$ for $\nu\in\{\lambda,\mu\}$ that appears in (4.125) into four summands. To this end, we utilize the Helmholtz decomposition (see Section A.4) to split the electric field $\mathbf{E}$ according to

$$\mathbf{E}=\mathsf{P}_\|\mathbf{E}+\mathsf{P}_\perp\mathbf{E}=\mathbf{E}^\|+\mathbf{E}^\perp. \tag{4.126}$$

From (4.84) we get that the curl-free part of the series $\{\mathbf{E}_\nu\}_{\nu\ge1}$ satisfies the relation

$$\forall\,\nu\ge1:\qquad \mathbf{E}^\|_\nu=-\mathsf{S}^\nu\mathsf{P}_\|\mathsf{G}\boldsymbol{a}_\nu\qquad\text{in }[0,T]\times\mathbb{R}^3. \tag{4.127}$$

Since for all $\mathbf{E}\in L^2(Q_T)$ we have $\big\|\mathsf{S}^\nu\mathbf{E}-\mathbf{E}\big\|_{L^2(Q_T)}\longrightarrow 0$ as $\nu\to\infty$ and for all $\nu\ge1$ we have $(\mathsf{S}^\nu)^*=\mathsf{S}^\nu$ (see Lemma A.3.13 (iv)) we may infer the relation

$$\mathbf{E}^\|_\infty=-\mathsf{P}_\|\mathsf{G}\boldsymbol{a}_\infty\qquad\text{in }[0,T]\times\mathbb{R}^3 \tag{4.128}$$

for the weak limits $\mathbf{E}_\infty$ and $\boldsymbol{a}_\infty$ by involving Lemma A.1.4. This yields the following decomposition of the terms $\mathbf{E}_\nu-\mathbf{E}_\infty$ for $\nu\in\{\lambda,\mu\}$

$$\mathbf{E}_\nu-\mathbf{E}_\infty=\big(\mathbf{E}^\perp_\nu-\mathbf{E}^\perp_\infty\big)-\mathsf{S}^\nu\mathsf{P}_\|\mathsf{G}(\boldsymbol{a}_\nu-\boldsymbol{a}_\infty)+\big(\mathsf{Id}-\mathsf{S}^\nu\big)\mathsf{P}_\|\mathsf{G}\boldsymbol{a}_\infty. \tag{4.129}$$

For each of the summands from (4.125), i.e. for $\nu=\lambda$ and for $\nu=\mu$, we thus get the following preliminary decomposition

$$\int_{\Omega_t} e^{-2b}f_1(\boldsymbol{a}_\nu)\,\mathsf{G}^*\big(\mathbf{E}_\nu-\mathbf{E}_\infty\big)\cdot\big(\boldsymbol{a}_\lambda-\boldsymbol{a}_\mu\big)\,dx\,ds=$$
$$\int_{\Omega_t} e^{-2b}f_1(\boldsymbol{a}_\nu)\,\mathsf{G}^*\big(\mathbf{E}^\perp_\nu-\mathbf{E}^\perp_\infty\big)\cdot\big(\boldsymbol{a}_\lambda-\boldsymbol{a}_\mu\big)\,dx\,ds \tag{4.130a}$$
$$+\int_{\Omega_t} e^{-2b}f_1(\boldsymbol{a}_\nu)\,\mathsf{G}^*\,\mathsf{S}^\nu\mathsf{P}_\|\big(\mathsf{G}(\boldsymbol{a}_\nu-\boldsymbol{a}_\infty)\big)\cdot\big(\boldsymbol{a}_\lambda-\boldsymbol{a}_\mu\big)\,dx\,ds \tag{4.130b}$$
$$+\int_{\Omega_t} e^{-2b}f_1(\boldsymbol{a}_\nu)\,\mathsf{G}^*\big(\mathsf{Id}-\mathsf{S}^\nu\big)\mathsf{P}_\|\mathsf{G}\boldsymbol{a}_\infty\cdot\big(\boldsymbol{a}_\lambda-\boldsymbol{a}_\mu\big)\,dx\,ds. \tag{4.130c}$$

We aim to apply a Gronwall argument to the difference of the term (4.130b) for $\nu = \lambda$ and for $\nu = \mu$. To this end, we need to put the factor e^{-b} in front of $(\boldsymbol{a}_\nu - \boldsymbol{a}_\infty)$. In order to perform the necessary commutations rigorously, we introduce the following notation.

Declaration 4.3.8. *For a given domain $\Omega \subseteq \mathbb{R}^3$ and a given Banach space X we denote by $\mathcal{F}(\Omega, \mathbb{R})$ the set of all functions $\varphi : \Omega \longrightarrow \mathbb{R}$ and by $\mathcal{F}(\Omega, X)$ the set of all functions $u : \Omega \longrightarrow X$. For a fixed function $\varphi \in \mathcal{F}(\Omega, \mathbb{R})$ we define the operator M_φ by*[17]

$$\mathsf{M}_\varphi : \begin{cases} \mathcal{F}(\Omega, X) \longrightarrow \mathcal{F}(\Omega, X) \\ \qquad\quad u \longmapsto \varphi \cdot u. \end{cases} \tag{4.131}$$

Here the $\cdot$ denotes the multiplication of a scalar.

With this notation, for all $p \in [1, \infty]$ and for all $t \in [0, T]$ the operator

$$\mathsf{M}_{e^{-b(\cdot,t)}} : \begin{cases} L^p(\Omega_{\mathrm{act}}) \longrightarrow L^p(\Omega_{\mathrm{act}}) \\ \qquad u(\cdot) \longmapsto e^{-b(\cdot,t)}\, u(\cdot) \end{cases} \tag{4.132}$$

is well defined, since for all $t \in [0, T]$ we have $e^{-b(\cdot,t)} \in L^p(\Omega_{\mathrm{act}})$ for all $p \in [1, \infty]$. We denote with $\widetilde{\mathsf{M}}_{e^{-b(\cdot,t)}}$ the corresponding operator

$$\widetilde{\mathsf{M}}_{e^{-b(\cdot,t)}} : \begin{cases} L^p(\mathbb{R}^3) \longrightarrow L^p(\mathbb{R}^3) \\ \qquad u(\cdot) \longmapsto e^{-b(\cdot,t)}\, u(\cdot). \end{cases} \tag{4.133}$$

Clearly, $\widetilde{\mathsf{M}}_{e^{-b(\cdot,t)}}$ is well defined for all $p \in [1, \infty]$ and for all $t \in [0, T]$. In particular, we have that with arbitrary $p \in [1, \infty]$ for all $\mathbf{U} \in L^p(\mathbb{R}^3, \mathbb{R}^3_{\mathbf{E}})$ and for all $\boldsymbol{v} \in L^p(\Omega_{\mathrm{act}}, \mathbb{R}^3_{\boldsymbol{a}})$ it holds

$$\forall\, t \in [0, T] : \quad \mathsf{M}_{e^{-b(t)}} \mathsf{G}^* \mathbf{U} = \mathsf{G}^* \widetilde{\mathsf{M}}_{e^{-b(\cdot,t)}} \mathbf{U}, \qquad \widetilde{\mathsf{M}}_{e^{-b(\cdot,t)}} \mathsf{G} \boldsymbol{v} = \mathsf{G} \mathsf{M}_{e^{-b(t)}} \boldsymbol{v}. \tag{4.134}$$

Therefore, setting $\boldsymbol{v}_\nu(t) := \boldsymbol{a}_\nu(t) - \boldsymbol{a}_\infty(t)$ we get that for all $t \in [0, T]$ it holds

$$\begin{aligned} e^{-2b(t)} \mathsf{G}^* \mathsf{S}^\nu \mathsf{P}_\| \big(\mathsf{G} \boldsymbol{v}_\nu(t)\big) &= e^{-b(t)} \mathsf{G}^* \widetilde{\mathsf{M}}_{e^{-b(\cdot,t)}} \mathsf{S}^\nu \mathsf{P}_\| \mathsf{G} \boldsymbol{v}_\nu(t) \\ &= e^{-b(t)} \mathsf{G}^* \big[\widetilde{\mathsf{M}}_{e^{-b(\cdot,t)}}, \mathsf{S}^\nu \mathsf{P}_\|\big] \mathsf{G} \boldsymbol{v}_\nu(t) + e^{-b(t)} \mathsf{G}^* \mathsf{S}^\nu \mathsf{P}_\| \widetilde{\mathsf{M}}_{e^{-b(\cdot,t)}} \mathsf{G} \boldsymbol{v}_\nu(t) \\ &= e^{-b(t)} \mathsf{G}^* \big[\widetilde{\mathsf{M}}_{e^{-b(\cdot,t)}}, \mathsf{S}^\nu \mathsf{P}_\|\big] \mathsf{G} \boldsymbol{v}_\nu(t) + e^{-b(t)} \mathsf{G}^* \mathsf{S}^\nu \mathsf{P}_\| \mathsf{G} \big(e^{-b(t)} \boldsymbol{v}_\nu(t)\big). \end{aligned} \tag{4.135}$$

Inserting this into the preliminary decomposition (4.130) yields the desired decomposition for each of the summands (4.125a), (4.125b), i.e. for $\nu = \lambda$ and for $\nu = \mu$ we have

[17] In this generality, the definition is formal. Of course, for concrete examples of functions φ and u one has to ensure that the object $\varphi \cdot u$ is well defined.

$$
\begin{aligned}
\int_{\Omega_t} & e^{-2b} f_1(\boldsymbol{a}_\nu)\, \mathsf{G}^*\big(\mathbf{E}_\nu - \mathbf{E}_\infty\big) \cdot \big(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big)\, dx\, ds = \\
& \int_{\Omega_t} e^{-2b} f_1(\boldsymbol{a}_\nu)\, \mathsf{G}^*\big(\mathbf{E}_\nu^\perp - \mathbf{E}_\infty^\perp\big) \cdot \big(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big)\, dx\, ds && \text{(4.136a)} \\
& + \int_{\Omega_t} e^{-b} f_1(\boldsymbol{a}_\nu)\, \mathsf{G}^*\, \mathsf{S}^\nu \mathsf{P}_\parallel \mathsf{G}\big(e^{-b}(\boldsymbol{a}_\nu - \boldsymbol{a}_\infty)\big) \cdot \big(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big)\, dx\, ds && \text{(4.136b)} \\
& + \int_{\Omega_t} e^{-b} f_1(\boldsymbol{a}_\nu)\, \mathsf{G}^*\big[\widetilde{\mathsf{M}}_{e^{-b}}, \mathsf{S}^\nu \mathsf{P}_\parallel\big]\big(\mathsf{G}(\boldsymbol{a}_\nu - \boldsymbol{a}_\infty)\big) \cdot \big(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big)\, dx\, ds && \text{(4.136c)} \\
& + \int_{\Omega_t} e^{-2b} f_1(\boldsymbol{a}_\nu)\, \mathsf{G}^*\big(\mathsf{Id} - \mathsf{S}^\nu\big)\mathsf{P}_\parallel \mathsf{G}\boldsymbol{a}_\infty \cdot \big(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big)\, dx\, ds. && \text{(4.136d)}
\end{aligned}
$$

In the following, we will successively discuss each of the above terms. Since the key difficulty of the whole proof is the discussions of the terms in (4.136a) and (4.136c), these discussions are postponed to the end of this section. The results are given in the propositions below.

Discussion of the terms (4.136d)
Since $\boldsymbol{a}_\infty$ is a weak $L^2(\Omega_T)$-limit (see (4.104b)), we have $\boldsymbol{a}_\infty \in L^2(\Omega_T)$. Thus, we have $\mathsf{P}_\parallel \mathsf{G}\boldsymbol{a}_\infty \in L^2(Q_T)$. We define the function $g_\nu := f_1(\boldsymbol{a}_\nu)\, \mathsf{G}^*\big(\mathsf{Id} - \mathsf{S}^\nu\big)\mathsf{P}_\parallel \mathsf{G}\boldsymbol{a}_\infty \in L^2(\Omega_T; \mathbb{R}^3_{\boldsymbol{a}})$. Then, the convergence $\big\|(\mathsf{S}^\nu - \mathsf{Id})\mathsf{P}_\parallel \mathsf{G}\boldsymbol{a}_\infty\big\|_{L^2(Q_T)} \longrightarrow 0$ (see Lemma A.3.13 (iv)), the bound $\big\|\mathsf{G}^*\big\|_{\mathcal{L}(L^2(\mathbb{R}^3), L^2(\Omega_{\mathrm{act}}))} \le C_{\mathsf{G}}$ from (4.7) and the bound $\sup_{\nu \ge 1} \|f_1(\boldsymbol{a}_\nu)\|_{L^\infty(Q_T)} \le C_f$ from (4.39) yield the strong convergence $g_\nu \longrightarrow 0$ in $L^2(\Omega_T; \mathbb{R}^3_{\boldsymbol{a}})$. The boundedness of the family $\{\boldsymbol{a}_\lambda\}_{\lambda \ge 1}$ in $L^2(\Omega_T)$ thus yields the convergence

$$
\lim_{\nu \to \infty} \int_{\Omega_t} e^{-b} f_1(\boldsymbol{a}_\nu)\, \mathsf{G}^*\big(\mathsf{Id} - \mathsf{S}^\nu\big)\mathsf{P}_\parallel \mathsf{G}\boldsymbol{a}_\infty \cdot \big(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big)\, dx\, ds = 0. \tag{4.137}
$$

Proposition 4.3.9 (**Discussion of the terms** (4.136a))**.** *For all $T > 0$ and for all $\delta > 0$, there exists $N = N(T, \delta) \ge 1$ such that for all λ, $\mu \ge N$ and for all $t \in [0, T]$ we have*

$$
\left| \int_{\Omega_t} e^{-2b} f_1(\boldsymbol{a}_\nu)\, \mathsf{G}^*\big(\mathbf{E}_\nu^\perp - \mathbf{E}_\infty^\perp\big) \cdot \big(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big)\, dx\, ds \right| \le \delta. \tag{4.138}
$$

Proposition 4.3.10 (**Discussion of the terms** (4.136c))**.** *For all $T > 0$ and for all $\delta > 0$, there exists $N = N(T, \delta) \ge 1$ such that for all λ, $\mu \ge N$ and for all $t \in [0, T]$ we have*

$$
\left| \int_{\Omega_t} e^{-b} f_1(\boldsymbol{a}_\nu)\, \mathsf{G}^*\big[\widetilde{\mathsf{M}}_{e^{-b}}, \mathsf{S}^\nu \mathsf{P}_\parallel\big]\big(\mathsf{G}(\boldsymbol{a}_\nu - \boldsymbol{a}_\infty)\big) \cdot \big(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big)\, dx\, ds \right| \le \delta. \tag{4.139}
$$

Discussion of the terms (4.136b)
Last but not least, due to Young's inequality we have the following estimate for the terms in (4.136b)

$$\begin{aligned}\int_{\Omega_t} & e^{-b} f_1(\boldsymbol{a}_\nu)\, \mathsf{G}^*\, \mathsf{S}^\nu \mathsf{P}_\parallel \mathsf{G}\big(e^{-b}(\boldsymbol{a}_\nu - \boldsymbol{a}_\infty)\big) \cdot \big(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big)\, dx\, ds \\ & \leq C_f \int_0^t \big\| \mathsf{G}^*\, \mathsf{S}^\nu \mathsf{P}_\parallel \mathsf{G}\big(e^{-b}(\boldsymbol{a}_\nu - \boldsymbol{a}_\infty)\big)\big\|^2_{L^2(\Omega_{\mathrm{act}})} + \big\| e^{-b}(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu)\big\|^2_{L^2(\Omega_{\mathrm{act}})}\, ds. \quad (4.140)\end{aligned}$$

Due to the estimate $\|\mathsf{G}^*\, \mathsf{S}^\nu \mathsf{P}_\parallel\, \mathsf{G}\|_{\mathcal{L}(L^2(\Omega_{\mathrm{act}}), L^2(\Omega_{\mathrm{act}}))} \leq C_\mathsf{G}^2$ we get

$$\leq C \int_0^t \big\| \big(e^{-b}(\boldsymbol{a}_\nu - \boldsymbol{a}_\infty)\big)\big\|^2_{L^2(\Omega_{\mathrm{act}})} + \big\| e^{-b}(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu)\big\|^2_{L^2(\Omega_{\mathrm{act}})}\, ds \quad (4.141)$$

for another constant C depending on $\{C_f, C_\mathsf{G}\}$.

Next, we study the difference of the terms in (4.141) for $\nu = \lambda$ and $\nu = \mu$. When $\nu = \mu$, we replace $\boldsymbol{a}_\mu - \boldsymbol{a}_\infty$ with $(\boldsymbol{a}_\mu - \boldsymbol{a}_\lambda) + (\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty)$ to achieve

$$\begin{aligned}\int_0^t & \big\| e^{-b}(\boldsymbol{a}_\mu - \boldsymbol{a}_\lambda) + e^{-b}(\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty)\big\|^2_{L^2(\Omega_{\mathrm{act}})} + \big\| e^{-b}(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu)\big\|^2_{L^2(\Omega_{\mathrm{act}})}\, ds \\ & \leq \int_0^t \big\| e^{-b}(\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty)\big\|^2_{L^2(\Omega_{\mathrm{act}})} + 2\,\big\| e^{-b}(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu)\big\|^2_{L^2(\Omega_{\mathrm{act}})}\, ds. \quad (4.142)\end{aligned}$$

Thus, utilizing (4.141) and (4.142) we can estimate the difference of the terms (4.136b) for $\nu = \lambda$ and $\nu = \mu$ by

$$\begin{aligned}\int_{\Omega_t} & e^{-b} f_1(\boldsymbol{a}_\lambda)\, \mathsf{G}^*\, \mathsf{S}^\lambda \mathsf{P}_\parallel \big(e^{-b}\mathsf{G}(\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty)\big) \cdot \big(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big)\, dx\, ds \\ & - \int_{\Omega_t} e^{-b} f_1(\boldsymbol{a}_\mu)\, \mathsf{G}^*\, \mathsf{S}^\mu \mathsf{P}_\parallel \big(e^{-b}\mathsf{G}(\boldsymbol{a}_\mu - \boldsymbol{a}_\infty)\big) \cdot \big(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big)\, dx\, ds \\ & \leq C \int_0^t \big\| e^{-b}(\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty)\big\|^2_{L^2(\Omega_{\mathrm{act}})} + \big\| e^{-b}(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu)\big\|^2_{L^2(\Omega_{\mathrm{act}})}\, ds \quad (4.143)\end{aligned}$$

where C is yet another constant depending on $\{C_f, C_\mathsf{G}\}$. Plugging decomposition (4.136) and the estimates (4.137), (4.138), (4.139), (4.143) into estimate (4.125) yields that for all $\delta > 0$, there exists $N = N(T, \delta) \geq 1$ such that for all λ, $\mu \geq N$ and for all $t \in [0, T]$ we have

$$\begin{aligned}& \big\| e^{-b(t)}\big(\boldsymbol{a}_\lambda(t) - \boldsymbol{a}_\mu(t)\big)\big\|^2_{L^2(\Omega_{\mathrm{act}})} \leq \\ & \delta + C \int_0^t \big\| e^{-b(s)}\big(\boldsymbol{a}_\lambda(s) - \boldsymbol{a}_\infty(s)\big)\big\|^2_{L^2(\Omega_{\mathrm{act}})}\, ds + C \int_0^t \big\| e^{-b(s)}\big(\boldsymbol{a}_\lambda(s) - \boldsymbol{a}_\mu(s)\big)\big\|^2_{L^2(\Omega_{\mathrm{act}})}\, ds.\end{aligned}$$

Using Gronwall's lemma (Theorem A.1.9), we get the estimate

$$\left\|e^{-b(t)}\big(\boldsymbol{a}_\lambda(t)-\boldsymbol{a}_\mu(t)\big)\right\|^2_{L^2(\Omega_{\mathrm{act}})} \le C_{\mathrm{cru}}\cdot\Big(\delta+\int_0^t \left\|e^{-b(s)}\big(\boldsymbol{a}_\lambda(s)-\boldsymbol{a}_\infty(s)\big)\right\|^2_{L^2(\Omega_{\mathrm{act}})}\,ds\Big) \tag{4.144}$$

with $C_{\mathrm{cru}} = C_{\mathrm{cru}}(T, C_f, C_{\mathsf{G}})$. This proves Lemma 4.3.7.

Proof of Proposition 4.3.9

The proof of this proposition is based on results from compensated compactness[18]. We begin with some technical preparations and write the integrand of (4.136a) as

$$e^{-2b} f_1(\boldsymbol{a}_\nu)\,\mathsf{G}^*\big(\mathbf{E}^\perp_\nu-\mathbf{E}^\perp_\infty\big)\cdot\big(\boldsymbol{a}_\lambda-\boldsymbol{a}_\mu\big) = e^{-2b}\boldsymbol{q}(\boldsymbol{a}_\lambda,\boldsymbol{a}_\mu)\cdot\mathsf{G}^*\big(\mathbf{E}^\perp_\nu-\mathbf{E}^\perp_\infty\big). \tag{4.145}$$

Here, $\boldsymbol{q}\in L^p(\Omega_T)$ for all $p\in[2,\infty]$ denotes the $\mathbb{R}^3_{\boldsymbol{a}}$-valued function $\boldsymbol{q}(\boldsymbol{a}_\lambda,\boldsymbol{a}_\mu) = f_1(\boldsymbol{a}_\nu)^T\,(\boldsymbol{a}_\lambda-\boldsymbol{a}_\mu)$. In view of (4.134) we have[19]

$$\begin{aligned}\int_{\Omega_t} e^{-2b}\boldsymbol{q}(\boldsymbol{a}_\lambda,\boldsymbol{a}_\mu)\cdot\mathsf{G}^*\big(\mathbf{E}^\perp_\nu-\mathbf{E}^\perp_\infty\big)\,dx\,ds &= \int_{Q_t}\mathsf{G}\mathsf{M}_{e^{-2b}}\boldsymbol{q}(\boldsymbol{a}_\lambda,\boldsymbol{a}_\mu)\cdot\big(\mathbf{E}^\perp_\nu-\mathbf{E}^\perp_\infty\big)\,dx\,ds\\ &= \int_{\Omega_t} e^{-2b}\mathsf{G}\boldsymbol{q}(\boldsymbol{a}_\lambda,\boldsymbol{a}_\mu)\cdot\big(\mathbf{E}^\perp_\nu-\mathbf{E}^\perp_\infty\big)\,dx\,ds.\end{aligned}$$

Furthermore, it holds $\sup_{\lambda,\mu\ge1}\left\|\boldsymbol{q}(\boldsymbol{a}_\lambda,\boldsymbol{a}_\mu)\right\|_{L^\infty(\Omega_T)}\le 2\,C_bC_f$. Therefore, we have

$$\sup_{\lambda,\mu\ge1}\left\|\mathsf{G}\boldsymbol{q}(\boldsymbol{a}_\lambda,\boldsymbol{a}_\mu)\cdot\big(\mathbf{E}^\perp_\nu-\mathbf{E}^\perp_\infty\big)\right\|_{L^2(\Omega_T)}\le 4\,C_{\mathsf{G}}C_bC_fC_{\mathbf{E}} =: C_{\boldsymbol{q}}. \tag{4.146}$$

For a given $\delta>0$, we choose an L^2-approximation $\psi\in C^\infty_c(\Omega_T)$ of the function e^{-2b} satisfying $\|e^{-2b}-\psi\|_{L^2(\Omega_T)}\le\delta/(2C_{\boldsymbol{q}})$. With this, we get that for all $t\in[0,T]$ it holds

$$\begin{aligned}&\big|(4.136\mathrm{a})\big| = \left|\int_{\Omega_t} e^{-2b}\mathsf{G}\boldsymbol{q}(\boldsymbol{a}_\lambda,\boldsymbol{a}_\mu)\cdot\big(\mathbf{E}^\perp_\nu-\mathbf{E}^\perp_\infty\big)\,dx\,ds\right|\\ &=\left|\int_{\Omega_t}\Big(\big(e^{-2b}-\psi\big)\mathsf{G}\boldsymbol{q}(\boldsymbol{a}_\lambda,\boldsymbol{a}_\mu)\cdot\big(\mathbf{E}^\perp_\nu-\mathbf{E}^\perp_\infty\big)+\psi\,\mathsf{G}\boldsymbol{q}(\boldsymbol{a}_\lambda,\boldsymbol{a}_\mu)\cdot\big(\mathbf{E}^\perp_\nu-\mathbf{E}^\perp_\infty\big)\Big)\,dx\,ds\right|\\ &\le\frac{\delta}{2C_{\boldsymbol{q}}}\left\|\mathsf{G}\boldsymbol{q}(\boldsymbol{a}_\lambda,\boldsymbol{a}_\mu)\cdot\big(\mathbf{E}^\perp_\nu-\mathbf{E}^\perp_\infty\big)\right\|_{L^2(\Omega_T)}+\left|\int_{\Omega_t}\psi\,\mathsf{G}\boldsymbol{q}(\boldsymbol{a}_\lambda,\boldsymbol{a}_\mu)\cdot\big(\mathbf{E}^\perp_\nu-\mathbf{E}^\perp_\infty\big)\,dx\,ds\right|\\ &\le\frac{\delta}{2}+\left|\int_{\Omega_t}\psi\,\mathsf{G}\boldsymbol{q}(\boldsymbol{a}_\lambda,\boldsymbol{a}_\mu)\cdot\big(\mathbf{E}^\perp_\nu-\mathbf{E}^\perp_\infty\big)\,dx\,ds\right|.\end{aligned} \tag{4.147}$$

The following argumentation is valid for all $t\in[0,T]$, we thus fix $t\in[0,T]$. Moreover, we will also denote the restrictions of $\mathsf{G}\boldsymbol{q}(\boldsymbol{a}_\lambda,\boldsymbol{a}_\mu)$ and $\big(\mathbf{E}^\perp_\nu-\mathbf{E}^\perp_\infty\big)$ to the set Ω_t with $\mathsf{G}\boldsymbol{q}(\boldsymbol{a}_\lambda,\boldsymbol{a}_\mu)$ and $\big(\mathbf{E}^\perp_\nu-\mathbf{E}^\perp_\infty\big)$, respectively.

[18] The precise result, Theorem A.6.2, is proved in [Gér91].

[19] In the last equality we use the fact that it makes no difference whether we integrate over Ω_t or Q_t.

First, we note that for all $t \in [0,T]$ the set $\{\mathsf{G}\boldsymbol{q}(\boldsymbol{a}_\lambda, \boldsymbol{a}_\mu)\}_{\lambda,\mu\geq 1}$ is bounded in $L^2(\Omega_t)$. Thus, there exist a subsequence $\{\mathsf{G}\boldsymbol{q}(\boldsymbol{a}_{\lambda_k}, \boldsymbol{a}_{\mu_k})\}_{k\in\mathbb{N}} \subset \{\mathsf{G}\boldsymbol{q}(\boldsymbol{a}_\lambda, \boldsymbol{a}_\mu)\}_{\lambda,\mu\geq 1}$ and a function $\widetilde{\boldsymbol{q}}_\infty \in L^2(\Omega_t)$ satisfying

$$\lim_{k\to\infty} \mathsf{G}\boldsymbol{q}(\boldsymbol{a}_{\lambda_k}, \boldsymbol{a}_{\mu_k}) \longrightarrow \widetilde{\boldsymbol{q}}_\infty \qquad \text{weakly in } L^2(\Omega_t). \tag{4.148}$$

Next, we show that for the set $\{\mathsf{G}\boldsymbol{q}(\boldsymbol{a}_\lambda, \boldsymbol{a}_\mu)\}_{\lambda,\mu\geq 1}$ we have that

$$\big\{\partial_t \mathsf{G}\boldsymbol{q}(\boldsymbol{a}_\lambda, \boldsymbol{a}_\mu)\big\}_{\lambda,\mu\geq 1} \text{ is relatively compact in } H^{-1}(\Omega_t). \tag{4.149}$$

To see this, note that the operator G does not depend on t and consider the estimate

$$\begin{aligned}
\big\|\partial_t \boldsymbol{q}(\boldsymbol{a}_\lambda,\boldsymbol{a}_\mu)\big\|_{L^2(\Omega_t)} &\leq \big\|L_1|\partial_t \boldsymbol{a}_\nu|(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu)\big\|_{L^2(\Omega_t)} + C_f\big\|\partial_t(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu)\big\|_{L^2(\Omega_t)} \\
&\leq 2\,L_1 C_b \big\|f(\boldsymbol{a}_\nu, \mathsf{G}^*\mathbf{E}_\nu)\big\|_{L^2(\Omega_t)} + C_f\big\|f(\boldsymbol{a}_\lambda, \mathsf{G}^*\mathbf{E}_\lambda) - f(\boldsymbol{a}_\mu, \mathsf{G}^*\mathbf{E}_\mu)\big\|_{L^2(\Omega_t)} \\
&\leq C(C_g, L_1, C_b, C_f)\cdot \sup_{\nu\geq 1}\Big(\big\|\big(1+|\boldsymbol{a}_\nu|\big)\big(1+|\mathsf{G}^*\mathbf{E}_\nu|\big)\big\|_{L^2(\Omega_t)}\Big) \\
&\leq C\cdot\big(t + C_{\boldsymbol{a}} + C_{\mathbf{E}}\big)
\end{aligned} \tag{4.150}$$

with a constant C depending on $\{\Omega_{\text{act}}, C_{\mathsf{G}}, C_g, L_1, C_b, C_f\}$. Thus, we have shown that the set $\big\{\partial_t \mathsf{G}\boldsymbol{q}(\boldsymbol{a}_\lambda, \boldsymbol{a}_\mu)\big\}_{\lambda,\mu\geq 1}$ is bounded in $L^2(\Omega_t)$. Since Ω_t is open and bounded, the compactness of the embedding $L^2(\Omega_t) \hookrightarrow H^{-1}(\Omega_t)$ from Lemma A.6.4 yields that the set $\big\{\partial_t \mathsf{G}\boldsymbol{q}(\boldsymbol{a}_\lambda, \boldsymbol{a}_\mu)\big\}_{\lambda,\mu\geq 1}$ is relatively compact in $H^{-1}(\Omega_t)$.

On the other hand, for all $t \in [0,T]$, the corresponding[20] subsequence $\big\{\big(\mathbf{E}^\perp_{\nu_k} - \mathbf{E}^\perp_\infty\big)\big\}_{k\geq 1}$ satisfies

$$\big(\mathbf{E}^\perp_{\nu_k} - \mathbf{E}^\perp_\infty\big) \longrightarrow 0 \qquad \text{weakly in } L^2(\Omega_t) \qquad \text{as } k\to\infty. \tag{4.151}$$

Moreover, denoting with $\Box$ the d-dimensional d'Alembert operator defined by

$$\Box := \big(\partial_t^2 - \Delta\big)\mathsf{Id}_{d\times d} = \Big(\partial_t^2 - \sum \partial_{x_j}^2\Big)\mathsf{Id}_{d\times d}, \tag{4.152}$$

we show next that for the sequence $\big\{\big(\mathbf{E}^\perp_\nu - \mathbf{E}^\perp_\infty\big)\big\}_{\nu\geq 1}$ it holds that

$$\big\{\Box\,\big(\mathbf{E}^\perp_\nu - \mathbf{E}^\perp_\infty\big)\big\}_{\nu\geq 1} \text{ is relatively compact in } H^{-2}(\Omega_t). \tag{4.153}$$

To see this, note that the operators $\mathsf{P}_\perp$, curl, and ∂_t as operators on $C^1([0,T]; H^1(\mathbb{R}^3))$ commute. Therefore, for all $\nu \geq 1$ the functions $\mathbf{E}_\nu$ and $\mathbf{H}_\nu$ satisfy[21]

$$\partial_t \mathbf{E}^\perp_\nu - \operatorname{curl} \mathsf{P}_\perp \mathbf{H}_\nu = -\mathsf{P}_\perp \mathsf{S}^\nu\big[\mathsf{G}f(\boldsymbol{a}_\nu, \mathsf{G}^*\mathbf{E}_\nu)\big] \tag{4.154a}$$

$$\partial_t \mathsf{P}_\perp \mathbf{H}_\nu = -\operatorname{curl} \mathsf{P}_\perp \mathbf{E}_\nu. \tag{4.154b}$$

[20]Note that ν is a placeholder for either λ or μ. Therefore, ν_k is to be understood as a placeholder for either of the above subsequences λ_k or μ_k.

[21]Note that due to Lemma 4.3.1, for all $\nu \geq 1$ we actually have $\mathbf{E}_\nu, \mathbf{H}_\nu \in C^1([0,T]; H^1(\mathbb{R}^3))$.

Applying ∂_t to the first and curl to the second of the equations above yields[22]

$$\forall\, \nu \geq 1: \qquad \big(\partial_t^2 - \Delta\big)\mathbf{E}_\nu^\perp = -\mathsf{P}_\perp \mathsf{S}^\nu \big[\mathsf{G}\partial_t f(\boldsymbol{a}_\nu, \mathsf{G}^*\mathbf{E}_\nu)\big], \tag{4.155}$$

since also S^ν, $\mathsf{P}_\perp$ and ∂_t commute and since it holds

$$\operatorname{curl}\operatorname{curl}\mathsf{P}_\perp = \nabla \operatorname{div}\mathsf{P}_\perp - \Delta\mathsf{P}_\perp = -\Delta\mathsf{P}_\perp. \tag{4.156}$$

Obviously, we have $\|\mathsf{S}^\nu\|_{\mathcal{L}(H^{-1},H^{-1})} = 1$ and $\|\mathsf{P}_\perp\|_{\mathcal{L}(H^{-1},H^{-1})} = 1$. Recalling the definition of norm $\|\cdot\|_{H^{-1}(\Omega_t)}$ in (A.42) and using that G does not depend on t, we get

$$\begin{aligned}
&\big\|\mathsf{P}_\perp\mathsf{S}^\nu\big[\mathsf{G}\,\partial_t f(\boldsymbol{a}_\nu, \mathsf{G}^*\mathbf{E}_\nu)\big]\big\|_{H^{-1}(\Omega_t)} \\
&\qquad \leq \big\|\partial_t\big(\mathsf{G}f(\boldsymbol{a}_\nu, \mathsf{G}^*\mathbf{E}_\nu)\big)\big\|_{H^{-1}(\Omega_t)} = \sup_{\substack{\varphi\in H_0^1(\Omega_t)\\ \|\varphi\|_{H_0^1(\Omega_t)}=1}} \Big|\int_{\Omega_t}\partial_t\varphi\cdot\mathsf{G}f(\boldsymbol{a}_\nu, \mathsf{G}^*\mathbf{E}_\nu)\,dx\,ds\Big| \\
&\qquad \leq C_{\mathsf{G}}\big\|f(\boldsymbol{a}_\nu, \mathsf{G}^*\mathbf{E}_\nu)\big\|_{L^2(\Omega_t)} \leq C\cdot\big(1 + C_{\boldsymbol{a}} + C_{\mathbf{E}}\big).
\end{aligned} \tag{4.157}$$

Here, the constant C depends on $\{C_{\mathsf{G}}, \Omega_{\mathrm{act}}, C_g\}$. This means that the set $\big\{\Box\mathbf{E}_\nu^\perp\big\}_{\nu\geq 1}$ is bounded in $H^{-1}(\Omega_t)$, thus, relatively compact in $H^{-2}(\Omega_t)$ in view of Lemma A.6.4. In particular, since $\mathbf{E}_\infty^\perp \in L^2(\Omega_t)$, we have $\Box\mathbf{E}_\infty^\perp \in H^{-2}(\Omega_t)$. This implies that the set $\big\{\Box\big(\mathbf{E}_\nu^\perp - \mathbf{E}_\infty^\perp\big)\big\}_{\nu\geq 1}$ is relatively compact in $H^{-2}(\Omega_t)$.

In the following, we denote by

$$\mathcal{C}_{\partial_t} := \big\{(\xi,\tau) \in \mathbb{S}^{(d+1)-1} : \tau = 0\big\} \tag{4.158}$$

the characteristic variety of the differential operator ∂_t and by

$$\mathcal{C}_{\Box} := \big\{(\xi,\tau) \in \mathbb{S}^{(d+1)-1} : \tau^2 - |\xi|^2 = 0\big\} \tag{4.159}$$

the characteristic variety of the differential operator $\Box$. Next, let

m_1 denote the microlocal defect measure of the sequence $\big\{\mathsf{G}\boldsymbol{q}(\boldsymbol{a}_{\lambda_k}, \boldsymbol{a}_{\mu_k})\big\}_{k\in\mathbb{N}}$,
m_2 denote the microlocal defect measure of the sequence $\big\{\big(\mathbf{E}_{\nu_k}^\perp - \mathbf{E}_\infty^\perp\big)\big\}_{k\in\mathbb{N}}$.

Then, we get from Lemma A.6.1 that for

$$A := \mathcal{C}_{\partial_t} \times \mathbb{R}^{d+1} \qquad \text{and} \qquad B := \mathcal{C}_{\Box} \times \mathbb{R}^{d+1} \tag{4.160}$$

it holds $m_1(A^c) = 0$ and $m_2(B^c) = 0$. Clearly, we have $A \cap B = \emptyset$. This means m_1 and m_2 are mutually singular.

[22]This equation has to be understood in the sense of distributions in the space $H^{-1}(\mathbb{R}^3)$.

Therefore, Theorem A.6.2 yields the following convergence as $k \to \infty$

$$\forall\, \psi \in C_c^\infty(\Omega_t): \qquad \int_{\Omega_t} \psi\,\big(\mathsf{G}\boldsymbol{q}(\boldsymbol{a}_{\lambda_k}, \boldsymbol{a}_{\mu_k}) \cdot \big(\mathbf{E}^\perp_{\nu_k} - \mathbf{E}^\perp_\infty\big)\big)\, dx\, dt \longrightarrow 0. \tag{4.161}$$

We note that the convergence (4.161) holds for the full subsequence from Declaration 4.3.4. As argued in the proof of Lemma 3.4.5, every subsequence of the family $\{\mathsf{G}\boldsymbol{q}(\boldsymbol{a}_\lambda, \boldsymbol{a}_\mu)\}_{\lambda,\mu\geq 1}$ has a weakly convergent subsequence. The limits may differ, but the above convergence does not depend on these limits. Thus, the above convergence holds for the full subsequence.

In particular, (4.161) implies that for every $\delta > 0$, there exists $N = N(\delta, T) \geq 1$ such that for all λ, $\mu \geq N$ and for all $t \in [0, T]$ we have

$$\left| \int_{\Omega_t} \psi\,\big(\mathsf{G}\boldsymbol{q}(\boldsymbol{a}_\lambda, \boldsymbol{a}_\mu) \cdot \big(\mathbf{E}^\perp_\nu - \mathbf{E}^\perp_\infty\big)\big)\, dx\, dt \right| \leq \frac{\delta}{2}. \tag{4.162}$$

This together with estimate (4.147) yields estimate (4.138) and proves Proposition 4.3.9.

Proof of Proposition 4.3.10

We start by taking $T > 0$ as before and fixing an arbitrary small $\delta > 0$. Then, we estimate the integral in (4.139) by

$$\begin{aligned}
&\left| \int_{\Omega_t} e^{-b} f_1(\boldsymbol{a}_\nu)\, \mathsf{G}^*\big[\widetilde{\mathsf{M}}_{e^{-b}}, \mathsf{S}^\nu \mathsf{P}_\|\big]\big(\mathsf{G}(\boldsymbol{a}_\nu - \boldsymbol{a}_\infty)\big) \cdot \big(\boldsymbol{a}_\lambda - \boldsymbol{a}_\mu\big)\, dx\, ds \right| \\
&\leq C_f \Big\| \mathsf{G}^*\big[\widetilde{\mathsf{M}}_{e^{-b}}, \mathsf{S}^\nu \mathsf{P}_\|\big]\big(\mathsf{G}(\boldsymbol{a}_\nu - \boldsymbol{a}_\infty)\big) \Big\|_{L^2(\Omega_t)} \big\| \boldsymbol{a}_\lambda - \boldsymbol{a}_\mu \big\|_{L^2(\Omega_t)} \\
&\leq 2\, C_f C_{\boldsymbol{a}} C_{\mathsf{G}} \Big\| \big[\widetilde{\mathsf{M}}_{e^{-b}}, \mathsf{S}^\nu \mathsf{P}_\|\big]\big(\mathsf{G}(\boldsymbol{a}_\nu - \boldsymbol{a}_\infty)\big) \Big\|_{L^2(Q_t)}.
\end{aligned} \tag{4.163}$$

Next, we introduce the series of functions $\{\boldsymbol{u}_\nu\}_{\nu\geq 1} \subset (L^2 \cap L^\infty)(Q_T)$ defined by $\boldsymbol{u}_\nu := \mathsf{G}(\boldsymbol{a}_\nu - \boldsymbol{a}_\infty)$ and the constant $C_{\mathrm{cs}} := 2\, C_f C_{\boldsymbol{a}} C_{\mathsf{G}}$. For every $\psi \in C_c^\infty((0,T) \times \mathbb{R}^3)$ we have[23]

$$\begin{aligned}
(4.163) = C_{\mathrm{cs}} &\Big\| \big[\widetilde{\mathsf{M}}_{e^{-b}}, \mathsf{S}^\nu \mathsf{P}_\|\big]\, \boldsymbol{u}_\nu \Big\|_{L^2(Q_t)} \leq C_{\mathrm{cs}} \Big\| \big[\widetilde{\mathsf{M}}_{e^{-b}}, \mathsf{S}^\nu \mathsf{P}_\|\big]\, \boldsymbol{u}_\nu \Big\|_{L^2(Q_T)} \\
&\leq C_{\mathrm{cs}} \Big\| \big[\widetilde{\mathsf{M}}_{e^{-b}-\psi}, \mathsf{S}^\nu \mathsf{P}_\|\big]\, \boldsymbol{u}_\nu + \big[\mathsf{M}_\psi, \mathsf{S}^\nu \mathsf{P}_\|\big]\, \boldsymbol{u}_\nu \Big\|_{L^2(Q_T)} \\
&\leq C_{\mathrm{cs}} \Big\| \big[\widetilde{\mathsf{M}}_{e^{-b}-\psi}, \mathsf{S}^\nu \mathsf{P}_\|\big]\, \boldsymbol{u}_\nu \Big\|_{L^2(Q_T)} + C_{\mathrm{cs}} \Big\| \big[\mathsf{M}_\psi, \mathsf{S}^\nu \mathsf{P}_\|\big]\, \boldsymbol{u}_\nu \Big\|_{L^2(Q_T)}.
\end{aligned} \tag{4.164}$$

As noted in (4.110) we have $e^{-b} \in L^p(Q_T)$ for all $p \in [1, \infty]$. In particular, for every $\widetilde{\delta} > 0$ there exists a function $\psi \in C_c^\infty((0,T) \times \mathbb{R}^3)$ with $\|e^{-b} - \psi\|_{L^4(Q_T)} \leq \widetilde{\delta}$. Therefore,

[23]The operator $\widetilde{\mathsf{M}}_{e^{-b}-\psi} : L^p(\mathbb{R}^3) \longrightarrow L^p(\mathbb{R}^3)$ is defined for all $p \in [1, \infty]$ by $\widetilde{\mathsf{M}}_{e^{-b}-\psi}[u] := (e^{-b} - \psi) \cdot u$.

we may estimate the first summand from (4.164) in the following way

$$\begin{aligned}\Big\|\big[\widetilde{\mathsf{M}}_{e^{-b}-\psi},\mathsf{S}^\nu\mathsf{P}_\|\big]\,\boldsymbol{u}_\nu\Big\|_{L^2(Q_T)} &\le \big\|e^{-b}-\psi\big\|_{L^4(Q_T)}\|\mathsf{S}^\nu\mathsf{P}_\|\big\|_{\mathcal{L}(L^4,L^4)}\big\|\boldsymbol{u}_\nu\big\|_{L^4(Q_T)}\\ &\quad+\big\|\mathsf{S}^\nu\mathsf{P}_\|\big\|_{\mathcal{L}(L^2,L^2)}\big\|e^{-b}-\psi\big\|_{L^4(Q_T)}\big\|\boldsymbol{u}_\nu\big\|_{L^4(Q_T)}\\ &\le 2\,\widetilde{\delta}\,\|\mathsf{S}^\nu\mathsf{P}_\|\big\|_{\mathcal{L}(L^4,L^4)}\big\|\boldsymbol{u}_\nu\big\|_{L^4(Q_T)}.\end{aligned}\tag{4.165}$$

For all $\nu\ge 1$ we have $\operatorname{supp}(\boldsymbol{u}_\nu)\subseteq\Omega_T\subset Q_T$ and $\big|\Omega_T\big|<\infty$. Therefore[24], for all $\nu\ge1$ it holds $\boldsymbol{u}_\nu\in L^p(Q_T)$ for all $p\in[2,\infty]$. Furthermore, from Lemma A.1.12 we get that for some constant $C_\|>0$ we have $\|\mathsf{P}_\|\|_{\mathcal{L}(L^p,L^p)}\le C_\|\,p$. Furthermore, Lemma A.3.13 yields that for all $\nu\ge1$ it holds $\|\mathsf{S}^\nu\|_{\mathcal{L}(L^p,L^p)}\le\|\mathcal{F}(\chi)\|_{L^1}$ with our fixed cut-off function χ from Section 4.3.1. This means that all norms in the above estimate are finite. In particular, choosing

$$\widetilde{\delta}=\frac{\delta}{4\big(4\,C_\|\,\|\mathcal{F}(\chi)\|_{L^1}\big)}\tag{4.166}$$

we get the estimate

$$\Big\|\big[\widetilde{\mathsf{M}}_{e^{-b}-\psi},\mathsf{S}^\nu\mathsf{P}_\|\big]\,\boldsymbol{u}_\nu\Big\|_{L^2(Q_T)}\le\frac{\delta}{2}.\tag{4.167}$$

For the second summand in (4.164) we can perform the following split

$$\begin{aligned}&\Big\|\big[\mathsf{M}_\psi,\mathsf{S}^\nu\mathsf{P}_\|\big]\,\boldsymbol{u}_\nu\Big\|_{L^2(Q_T)}\\ &\qquad\le\Big\|\big[\mathsf{M}_\psi,\mathsf{S}^\nu\mathsf{P}_\|(\mathsf{Id}-\mathsf{S}^1)\big]\,\boldsymbol{u}_\nu\Big\|_{L^2(Q_T)}+\Big\|\big[\mathsf{M}_\psi,\mathsf{S}^1\mathsf{P}_\|\big]\,\boldsymbol{u}_\nu\Big\|_{L^2(Q_T)}.\end{aligned}\tag{4.168}$$

Next, we fix $t\in[0,T]$ and recall from Lemma 4.3.6 that $\boldsymbol{a}_\infty$ is continuous in time[25] and that for all $t\in[0,T]$ we have $\boldsymbol{a}_\nu(t)\longrightarrow\boldsymbol{a}_\infty(t)$ weakly in $L^2(\Omega_{\mathrm{act}})$. This implies that we have the convergence

$$\forall\,t\in[0,T]:\qquad \boldsymbol{u}_\nu(t)\longrightarrow 0\qquad\text{weakly in } L^2(\mathbb{R}^3).\tag{4.169}$$

Moreover, due to Lemma A.3.12 and Lemma A.3.13 (vii) the family of pseudo-differential operators $\big\{\mathsf{S}^\nu\mathsf{P}_\|(\mathsf{Id}-\mathsf{S}^1)\big\}_{\lambda\ge1}$ is defined via a bounded family of pure multiplier symbols of degree zero and for every $t\in[0,T]$ we have $\psi(t)\in C_c^\infty(\mathbb{R}^3)$. Therefore, taking into account Remark A.3.16, Theorem A.3.14 yields the pointwise convergence

$$\forall\,t\in[0,T]:\qquad\Big\|\big[\mathsf{M}_{\psi(t)},\mathsf{S}^\nu\mathsf{P}_\|(\mathsf{Id}-\mathsf{S}^1)\big]\,\boldsymbol{u}_\nu(t)\Big\|_{L^2(\mathbb{R}^3)}\longrightarrow 0\qquad\text{as }\nu\to\infty.\tag{4.170}$$

Strictly seen, for every $t\in[0,T]$ the operator $\big[\mathsf{M}_{\psi(t)},\mathsf{S}^1\mathsf{P}_\|\big]$ is not a pseudo-differential operator of any order, since every derivative of the symbol $\pi_\|$ of $\mathsf{P}_\|$ is singular for $\xi=0$.

[24] In fact, we do not need $\big|\Omega_T\big|<\infty$. Since for all $\nu\ge1$ we have $\boldsymbol{u}_\nu\in(L^2\cap L^\infty)(Q_T)$, we may infer from Lemma A.1.11 that for all $\nu\ge1$ it holds $\boldsymbol{u}_\nu\in L^p(Q_T)$ for all $p\in[2,\infty]$.

[25] Strictly speaking, the equivalence class $\boldsymbol{a}_\infty$ has a representative $\widetilde{\boldsymbol{a}}_\infty\in C^0([0,T];L^2(\Omega_{\mathrm{act}}))$.

Nevertheless, for the operator $\mathsf{M}_{\psi(t)}$ we have $\|\mathsf{M}_{\psi(t)}\|_{\mathcal{L}(H^s,H^s)} \leq \|\psi(t)\|_{W^{s,\infty}}$ and involving the Bernstein type lemma, we get the following estimate for all $s \in \mathbb{R}$

$$\begin{aligned}\Big\|\big[\mathsf{M}_{\psi(t)}, \mathsf{S}^1\mathsf{P}_{\|}\big]\Big\|_{\mathcal{L}(H^s,H^{s+1})} &\leq 2\|\mathsf{M}_{\psi(t)}\|_{\mathcal{L}(H^s,H^s)}\,\|\mathsf{S}^1\|_{\mathcal{L}(H^s,H^{s+1})}\,\|\mathsf{P}_{\|}\|_{\mathcal{L}(H^s,H^s)} \\ &\leq 2\,C_{\mathrm{bsl}}^2\,\|\psi(t)\|_{W^{s,\infty}}. \end{aligned} \tag{4.171}$$

Thus, using the same arguments as in the proof of Theorem A.3.14 shows that we have the convergence

$$\forall\, t \in [0,T]: \qquad \Big\|\big[\mathsf{M}_{\psi(t)}, \mathsf{S}^1\mathsf{P}_{\|}\big]\,\boldsymbol{u}_\nu(t)\Big\|_{L^2(\mathbb{R}^3)} \longrightarrow 0 \qquad \text{as } \nu \to \infty. \tag{4.172}$$

Next, we define the family of functions $u_\nu \in L^1(0,T)$ given by

$$u_\nu(t) := \Big\|\big[\mathsf{M}_{\psi(t)}, \mathsf{S}^\nu\mathsf{P}_{\|}(\mathsf{Id}-\mathsf{S}^1)\big]\,\boldsymbol{u}_\nu(t)\Big\|^2_{L^2(\mathbb{R}^3)} + \Big\|\big[\mathsf{M}_{\psi(t)}, \mathsf{S}^1\mathsf{P}_{\|}\big]\,\boldsymbol{u}_\nu(t)\Big\|^2_{L^2(\mathbb{R}^3)}. \tag{4.173}$$

From (4.170) and (4.172) we get the pointwise convergence $u_\nu(t) \longrightarrow 0$ for all $t \in [0,T]$. Moreover, in view of (4.115) we have the bound

$$\forall\, t \in [0,T],\ \forall\, \nu \geq 1: \qquad |u_\nu(t)| \leq 8\big(\|\mathsf{G}^*\|_{L^\infty} C_{\boldsymbol{a}}^{\mathrm{uni}}\|\psi(t)\|_{L^\infty}\big)^2. \tag{4.174}$$

Thus, Lebesgue's dominated convergence theorem (see Theorem A.1.1) yields the convergence $u_\nu \longrightarrow 0$ strongly in $L^1(0,T)$ as $\nu \to \infty$. This implies

$$\Big\|\big[\mathsf{M}_\psi, \mathsf{S}^\nu\mathsf{P}_{\|}\big]\,\boldsymbol{u}_\nu\Big\|_{L^2(Q_T)} \longrightarrow 0 \qquad \text{as } \nu \to \infty \tag{4.175}$$

and finishes the proof of Proposition 4.3.10.

4.3.5. Strong Convergence of the Fields

In this section we show the convergence $\mathbf{E}_\lambda \longrightarrow \mathbf{E}_\infty$ strongly in $C^0([0,T];L^2(\mathbb{R}^3))$ and that $(\mathbf{E}_\infty, \mathbf{H}_\infty)$ is a distributional solution to the Maxwell part (4.70a)–(4.70b) of (4.70) in the sense of (4.49). For this proof we do not follow the lines of [JMR00a] (or [Dum05]), but base our arguments on Lemma A.1.3. First, we show the following result.

Lemma 4.3.11. *The triplet $(\mathbf{E}_\infty, \mathbf{H}_\infty, \boldsymbol{a}_\infty)$ is a distributional solution to the Maxwell part* (4.70a)–(4.70b) *of system* (4.70)*, i.e. for all test functions $\psi \in C_c^\infty([0,T)\times\mathbb{R}^3;\mathbb{R}^6)$, the triplet $(\mathbf{E}_\infty, \mathbf{H}_\infty, \boldsymbol{a}_\infty)$ satisfies*

$$\begin{aligned}&\int_0^T\!\!\int_{\mathbb{R}^3} \partial_t\psi(x,s)\cdot\big(\mathbf{E}_\infty(x,s),\mathbf{H}_\infty(x,s)\big)\,dx\,ds + \int_{\mathbb{R}^3}\psi(x,0)\cdot\big(\mathbf{E}_0(x),\mathbf{H}_0(x)\big)\,dx \\ &- \int_0^T\!\!\int_{\mathbb{R}^3} \mathsf{B}^*[\psi(x,s)]\cdot\big(\mathbf{E}_\infty(x,s),\mathbf{H}_\infty(x,s)\big)\,dx\,ds \\ &= \int_0^T\!\!\int_{\mathbb{R}^3}\psi(x,s)\cdot\Big(\mathsf{G}f\big(\boldsymbol{a}_\infty(x,s),\mathsf{G}^*\mathbf{E}_\infty(x,s)\big),0\Big)\,dx\,ds. \end{aligned} \tag{4.176}$$

Proof. Remark 4.3.2 yields that for all $\lambda \geq 1$ the triplet $\big(\mathbf{E}_\lambda, \mathbf{H}_\lambda, \boldsymbol{a}_\lambda\big)$ satisfies (4.93). Taking the difference of the equations (4.93) and (4.176) yields

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} \partial_t \psi \cdot \big(\mathbf{E}_\lambda - \mathbf{E}_\infty, \mathbf{H}_\lambda - \mathbf{H}_\infty\big)\, dx\, ds + \int_{\mathbb{R}^3} \psi(x,0) \cdot \big(\mathbf{E}_0^\lambda - \mathbf{E}_0, \mathbf{H}_0^\lambda - \mathbf{H}_0\big)\, dx \\
&+ \int_0^T \int_{\mathbb{R}^3} \mathsf{B}^*[\psi(x,s)] \cdot \big(\mathbf{E}_\lambda - \mathbf{E}_\infty, \mathbf{H}_\lambda - \mathbf{H}_\infty\big)\, dx\, ds \\
&= \int_0^T \int_{\mathbb{R}^3} \psi \cdot \Big(\mathsf{S}^\lambda \mathsf{G} f\big(\boldsymbol{a}_\lambda, \mathsf{G}^*\mathbf{E}_\lambda\big) - \mathsf{G} f\big(\boldsymbol{a}_\infty, \mathsf{G}^*\mathbf{E}_\infty\big), 0\Big)\, dx\, ds.
\end{aligned} \tag{4.177}$$

Obviously, writing $\psi = \big(\psi_1, \psi_2\big)$ we can split the right hand side of (4.177) according to

$$(4.177) = \int_0^T \int_{\mathbb{R}^3} \psi_1 \cdot \mathsf{S}^\lambda \mathsf{G}\Big(f\big(\boldsymbol{a}_\lambda, \mathsf{G}^*\mathbf{E}_\lambda\big) - f\big(\boldsymbol{a}_\infty, \mathsf{G}^*\mathbf{E}_\infty\big)\Big)\, dx\, ds \tag{4.178a}$$

$$+ \int_0^T \int_{\mathbb{R}^3} \psi_1 \cdot \big(\mathsf{S}^\lambda \mathsf{G} - \mathsf{G}\big)\big(f\big(\boldsymbol{a}_\infty, \mathsf{G}^*\mathbf{E}_\infty\big)\big)\, dx\, ds. \tag{4.178b}$$

Due to the weak convergence $\big(\mathbf{E}_\lambda, \mathbf{H}_\lambda\big) \longrightarrow \big(\mathbf{E}_\infty, \mathbf{H}_\infty\big)$ in $L^2(Q_T)$, the strong convergence $\big(\mathbf{E}_0^\lambda, \mathbf{H}_0^\lambda\big) \longrightarrow \big(\mathbf{E}_0, \mathbf{H}_0\big)$ in $L^2(\mathbb{R}^3)$ and since we have $\partial_t\psi, \mathsf{B}^*[\psi] \in L^2(Q_T)$, the left hand side of equation (4.177) converges to zero. From estimate (4.43) in Remark 4.1.6 we get that $f(\boldsymbol{a}_\infty, \mathsf{G}^*\mathbf{E}_\infty) \in L^2(\Omega_T)$, i. e. $\mathsf{G} f(\boldsymbol{a}_\infty, \mathsf{G}^*\mathbf{E}_\infty) \in L^2(Q_T)$. The convergence

$$\big\|\big(\mathsf{S}^\lambda \mathsf{G} - \mathsf{G}\big) f(\boldsymbol{a}_\infty, \mathsf{G}^*\mathbf{E}_\infty)\big\|_{L^2} = \big\|\big(\mathsf{S}^\lambda - \mathsf{Id}\big)\mathsf{G} f(\boldsymbol{a}_\infty, \mathsf{G}^*\mathbf{E}_\infty)\big\|_{L^2} \longrightarrow 0 \tag{4.179}$$

yields that the second summand (4.178b) tends to zero as $\lambda \to \infty$. Moreover, due to the Lipschitz estimates on f (see (4.40a), (4.40b) and (4.42)), we may estimate the first summand (4.178a) according to

$$\int_{Q_T} \psi_1 \cdot \mathsf{S}^\lambda \mathsf{G}\Big(f\big(\boldsymbol{a}_\lambda, \mathsf{G}^*\mathbf{E}_\lambda\big) - f\big(\boldsymbol{a}_\infty, \mathsf{G}^*\mathbf{E}_\infty\big)\Big)\, ds\, dx$$

$$\leq \int_{\Omega_T} \psi_1 \cdot C\Big(L_0\big|\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty\big| + L_1\big|\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty\big|\big|\mathsf{G}^*\mathbf{E}_\infty\big|\Big)\, ds\, dx \tag{4.180a}$$

$$+ \int_{Q_T} \psi_1 \cdot \mathsf{S}^\lambda \mathsf{G}\Big(f_1(\boldsymbol{a}_\lambda)\mathsf{G}^*(\mathbf{E}_\lambda - \mathbf{E}_\infty)\Big)\, ds\, dx \tag{4.180b}$$

with a constant C depending on C_G. The summand (4.180a) tends to zero due to the strong convergence $\boldsymbol{a}_\lambda \longrightarrow \boldsymbol{a}_\infty$ in $L^2(\Omega_T)$.

For all $\lambda \geq 1$ the operator S^λ linearly maps the space $L^2(\mathbb{R}^3; \mathbb{R}^3_\mathbf{E})$ into itself with $\|\mathsf{S}^\lambda\|_{\mathcal{L}(L^2,L^2)} \leq 1$. Furthermore, since $f_1(\boldsymbol{a}_\lambda) \in L^\infty(\Omega_T)$ is $\mathcal{L}(\mathbb{R}^3_{\boldsymbol{a}}, \mathbb{R}^3_{\boldsymbol{a}})$-valued, we can also interpret $f_1(\boldsymbol{a}_\lambda)$ as a linear operator with adjoint $(f_1(\boldsymbol{a}_\lambda))^T$ mapping $L^2(\Omega_T; \mathbb{R}^3_{\boldsymbol{a}})$ into itself. Recalling the definitions of G^* and G from (4.7)–(4.8), we get that

$$\mathsf{A}_\lambda := \mathsf{S}^\lambda \circ \mathsf{G} \circ f_1(\boldsymbol{a}_\lambda) \circ \mathsf{G}^* \tag{4.181}$$

defines a bounded family of linear operators on $L^2(Q_T; \mathbb{R}^3_\mathbf{E})$.

For the adjoints A_λ^* of these linear operators defined[26] by $\mathsf{A}_\lambda^* := (\mathsf{G}^*)^T \circ f_1(\boldsymbol{a}_\lambda)^T \circ \mathsf{G}^T \circ \mathsf{S}^\lambda$, the following convergence holds.

$$\forall\, \mathbf{F} \in L^2(Q_T): \quad \big\|(\mathsf{A}_\lambda^* - \mathsf{A}^*)\mathbf{F}\big\|_{L^2(Q_T)} \longrightarrow 0 \qquad \text{with} \qquad \mathsf{A}^* := (\mathsf{G}^*)^T \circ f_1(\boldsymbol{a}_\lambda)^T \circ \mathsf{G}^T. \tag{4.182}$$

This convergence easily follows from a short calculation involving Lemma A.1.3 by exploiting the strong convergence $\boldsymbol{a}_\lambda \longrightarrow \boldsymbol{a}_\infty$ in $L^2(\Omega_T)$ and the uniform L^∞-bound on $\boldsymbol{a}_\lambda$ from Lemma 4.3.3. With (4.182), we may infer from Lemma A.1.4 the convergence

$$\mathsf{A}_\lambda(\mathbf{E}_\lambda - \mathbf{E}_\infty) \longrightarrow 0 \qquad \text{weakly in } L^2(Q_T). \tag{4.183}$$

Therefore, also the second summand (4.180b) tends to zero. Thus, the summand (4.178a) tends to zero as $\lambda \to \infty$. This shows that in fact the triplet $\big(\mathbf{E}_\infty, \mathbf{H}_\infty, \boldsymbol{a}_\infty\big)$ satisfies (4.176) and proves Lemma 4.3.11. $\square$

We have shown that $\big(\mathbf{E}_\infty, \mathbf{H}_\infty\big)$ is a distributional solution to a symmetric[27] hyperbolic system with right hand side $\mathsf{G}f(\boldsymbol{a}_\infty, \mathbf{E}_\infty) \in L^2(Q_T)$ and initial condition $\big(\mathbf{E}_0, \mathbf{H}_0\big) \in L^2(\mathbb{R}^3)$. Lemma A.5.5 yields $\big(\mathbf{E}_\infty, \mathbf{H}_\infty\big) \in C^0([0,T]; L^2(\mathbb{R}^3))$. We end this section with showing the convergence

$$\big(\mathbf{E}_\lambda, \mathbf{H}_\lambda\big) \longrightarrow \big(\mathbf{E}_\infty, \mathbf{H}_\infty\big) \qquad \text{strongly in } C^0([0,T]; L^2(\mathbb{R}^3)) \qquad \text{as } \lambda \to \infty. \tag{4.184}$$

To this end, we note that for every $\lambda \geq 1$, the difference $U_\lambda := \big(\mathbf{E}_\lambda - \mathbf{E}_\infty, \mathbf{H}_\lambda - \mathbf{H}_\infty\big)$ solves a symmetric hyperbolic system with right hand side

$$f_\lambda := \mathsf{S}^\lambda \mathsf{G} f(\boldsymbol{a}_\lambda, \mathsf{G}^*\mathbf{E}_\lambda) - \mathsf{G} f(\boldsymbol{a}_\infty, \mathsf{G}^*\mathbf{E}_\infty) \in L^2(Q_T) \tag{4.185}$$

and initial datum $U_0^\lambda := \big(\mathbf{E}_0^\lambda - \mathbf{E}_0, \mathbf{H}_0^\lambda - \mathbf{H}_0\big) \in L^2(\mathbb{R}^3)$. In particular, Proposition A.5.2 yields that for all $t \in [0,T]$ the function U_λ satisfies the estimate

$$\big\|U_\lambda(t)\big\|^2_{L^2(\mathbb{R}^3)} \leq C\Big(\big\|U_0^\lambda\big\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \big\|f_\lambda(s)\big\|^2_{L^2(\mathbb{R}^3)}\, ds\Big). \tag{4.186}$$

For the right hand side f_λ we have the following estimate since f is Lipschitz continuous

$$\begin{aligned} f_\lambda &= \big(\mathsf{S}^\lambda - \mathsf{Id}\big)\mathsf{G} f(\boldsymbol{a}_\infty, \mathsf{G}^*\mathbf{E}_\infty) + \mathsf{S}^\lambda \mathsf{G}\big(f(\boldsymbol{a}_\lambda, \mathsf{G}^*\mathbf{E}_\lambda) - f(\boldsymbol{a}_\infty, \mathsf{G}^*\mathbf{E}_\infty)\big) \\ &\leq \big(\mathsf{S}^\lambda - \mathsf{Id}\big)\mathsf{G} f(\boldsymbol{a}_\infty, \mathsf{G}^*\mathbf{E}_\infty) \\ &\qquad + \mathsf{S}^\lambda \mathsf{G}\big(L_0\big|\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty\big| + L_1\big|\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty\big|\,\big|\mathsf{G}^*\mathbf{E}_\infty\big| + C_f\big|\mathsf{G}^*\mathbf{E}_\lambda - \mathsf{G}^*\mathbf{E}_\infty\big|\big). \end{aligned}$$

[26]The operator G^T is defined as G but with γ replaced by γ^* and the operator $(\mathsf{G}^*)^T$ is defined as G^* but with γ^* replaced by γ.

[27]The symmetry is due to setting $\epsilon_0 = 1 = \mu_0$. Achieving this symmetry was the reason for doing so.

Involving Young's inequality and introducing $Q_t := (0,t) \times \mathbb{R}^3$, this yields the following estimate for all $t \in [0,T]$

$$\begin{aligned}
\|f_\lambda\|^2_{L^2(Q_t)} &\leq 2\Big(\big\|(\mathsf{S}^\lambda - \mathsf{Id})\mathsf{G}f(\boldsymbol{a}_\infty, \mathsf{G}^*\mathbf{E}_\infty)\big\|^2_{L^2(Q_t)} \\
&\qquad + \big\|\mathsf{S}^\lambda\mathsf{G}\big(L_0|\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty| + L_1|\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty|\,|\mathsf{G}^*\mathbf{E}_\infty| + C_f|\mathsf{G}^*\mathbf{E}_\lambda - \mathsf{G}^*\mathbf{E}_\infty|\big)\big\|^2_{L^2(Q_t)}\Big) \\
&\leq C\Big(\big\|(\mathsf{S}^\lambda - \mathsf{Id})\mathsf{G}f(\boldsymbol{a}_\infty, \mathsf{G}^*\mathbf{E}_\infty)\big\|^2_{L^2(Q_t)} + \|\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty\|^2_{L^2(\Omega_t)} \\
&\qquad + \big\||\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty|\,|\mathsf{G}^*\mathbf{E}_\infty|\big\|^2_{L^2(\Omega_t)} + C_{\mathsf{G}}\|\mathbf{E}_\lambda - \mathbf{E}_\infty\|^2_{L^2(Q_t)}\Big)
\end{aligned} \tag{4.187}$$

with a constant C depending on $\{C_{\mathsf{G}}, L_0, L_1, C_f\}$. For the estimate on U_λ, this implies that for all $t \in [0,T]$ we have

$$\begin{aligned}
\|U_\lambda(t)\|^2_{L^2(\mathbb{R}^3)} &\leq C\Big(\|U_0^\lambda\|^2_{L^2} + \big\|(\mathsf{S}^\lambda - \mathsf{Id})\mathsf{G}f(\boldsymbol{a}_\infty, \mathsf{G}^*\mathbf{E}_\infty)\big\|^2_{L^2(Q_t)} + \|\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty\|^2_{L^2(\Omega_t)} \\
&\qquad + \big\||\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty|\,|\mathsf{G}^*\mathbf{E}_\infty|\big\|^2_{L^2(\Omega_t)} + \int_0^t \|U_\lambda(s)\|^2_{L^2(\mathbb{R}^3)}\,ds\Big).
\end{aligned} \tag{4.188}$$

Gronwall's lemma (Theorem A.1.9) thus yields that for all $t \in [0,T]$ we have

$$\begin{aligned}
\|U_\lambda(t)\|^2_{L^2(\mathbb{R}^3)} &\leq C\Big(\|U_0^\lambda\|^2_{L^2} + \big\|(\mathsf{S}^\lambda - \mathsf{Id})\mathsf{G}f(\boldsymbol{a}_\infty, \mathsf{G}^*\mathbf{E}_\infty)\big\|^2_{L^2(Q_T)} \\
&\qquad + \|\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty\|^2_{L^2(\Omega_T)} + \big\||\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty|\,|\mathsf{G}^*\mathbf{E}_\infty|\big\|^2_{L^2(\Omega_T)}\Big).
\end{aligned} \tag{4.189}$$

For $\lambda \to \infty$ the first summand in (4.189) tends to zero due to (4.103). Obviously we have

$$\mathsf{G}f(\boldsymbol{a}_\infty, \mathsf{G}^*\mathbf{E}_\infty) \in L^2(Q_T). \tag{4.190}$$

From Lemma A.3.13 (iv) we may therefore infer the convergence

$$\big\|(\mathsf{S}^\lambda - \mathsf{Id})\mathsf{G}f(\boldsymbol{a}_\infty, \mathsf{G}^*\mathbf{E}_\infty)\big\|_{L^2(Q_T)} \longrightarrow 0, \qquad \lambda \to \infty. \tag{4.191}$$

The convergence $\|\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty\|_{L^2(\Omega_T)} \longrightarrow 0$ is clear due to (4.124). By involving Lemma A.1.3, the strong convergence from (4.124) and the uniform L^∞-bound on $\boldsymbol{a}_\lambda$ from Lemma 4.3.3, we get that also the last summand in (4.189) tends to zero. This shows the convergence

$$\big(\mathbf{E}_\lambda, \mathbf{H}_\lambda\big) \longrightarrow \big(\mathbf{E}_\infty, \mathbf{H}_\infty\big) \qquad \text{strongly in } C^0([0,T]; L^2(\mathbb{R}^3)) \qquad \text{as } \lambda \to \infty. \tag{4.192}$$

4.3.6. Conclusion

We have shown that the triplet $\big(\mathbf{E}_\infty, \mathbf{H}_\infty, \boldsymbol{a}_\infty\big)$ satisfies (4.176) and that we have the regularity $\big(\mathbf{E}_\infty, \mathbf{H}_\infty\big) \in C^0([0,T]; L^2(\mathbb{R}^3))$. Thus, it remains to show that (A) this triplet solves the ODE (4.70c) and that (B) it holds $\big(\mathbf{E}_\infty, \mathbf{H}_\infty, \boldsymbol{a}_\infty\big) \in C^0([0,T]; \mathbf{L}_{\text{div}})$.

To prove (A), we define the following function for $(x,t) \in (0,T) \times \Omega_{\mathrm{act}}$

$$\boldsymbol{a}^\infty(x,t) := \boldsymbol{a}_0(x) + \int_0^t \Big(f_0\big(\boldsymbol{a}_\infty(x,s)\big) + f_1\big(\boldsymbol{a}_\infty(x,s)\big)\, \mathsf{G}^*\mathbf{E}_\infty(x,s)\Big)\, ds. \tag{4.193}$$

The Lipschitz estimates on f from (4.40a), (4.40b) and (4.42) yield the following estimate for all $\lambda \geq 1$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_{\mathrm{act}}} \big|\boldsymbol{a}_\lambda - \boldsymbol{a}^\infty\big|^2\, dx = \int_{\Omega_{\mathrm{act}}} \big(\boldsymbol{a}_\lambda - \boldsymbol{a}^\infty\big)\cdot \partial_t\big(\boldsymbol{a}_\lambda - \boldsymbol{a}^\infty\big)\, dx \\
&= \int_{\Omega_{\mathrm{act}}} \big(\boldsymbol{a}_\lambda - \boldsymbol{a}^\infty\big)\cdot \big(f_0(\boldsymbol{a}_\lambda) - f_0(\boldsymbol{a}_\infty) + f_1(\boldsymbol{a}_\lambda)\mathsf{G}^*\mathbf{E}_\lambda - f_1(\boldsymbol{a}_\infty)\mathsf{G}^*\mathbf{E}_\infty\big)\, dx \\
&\leq \int_{\Omega_{\mathrm{act}}} \big|\boldsymbol{a}_\lambda - \boldsymbol{a}^\infty\big| \Big(L_0|\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty| + L_1|\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty|\,|\mathsf{G}^*\mathbf{E}_\infty| + C_f|\mathsf{G}^*\mathbf{E}_\lambda - \mathsf{G}^*\mathbf{E}_\infty|\Big)\, dx.
\end{aligned}$$

Applying Young's inequality for each summand and integrating the resulting estimate over $(0,t)$ for an arbitrary $t \in (0,T)$ yields

$$\begin{aligned}
\big\|(\boldsymbol{a}_\lambda - \boldsymbol{a}^\infty)(t)\big\|^2_{L^2(\Omega_{\mathrm{act}})} \leq{}& \int_0^t \big\|(\boldsymbol{a}_\lambda - \boldsymbol{a}^\infty)(s)\big\|^2_{L^2(\Omega_{\mathrm{act}})}\, ds + L_0\big\|\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty\big\|^2_{L^2(\Omega_t)} \\
&+ L_1\big\||\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty|\,|\mathsf{G}^*\mathbf{E}_\infty|\big\|^2_{L^2(\Omega_t)} + C_f C_{\mathsf{G}}\big\|\mathbf{E}_\lambda - \mathbf{E}_\infty\big\|^2_{L^2(Q_t)}
\end{aligned}$$

with $Q_t := (0,t)\times\mathbb{R}^3$ and $\Omega_t := (0,t)\times\Omega_{\mathrm{act}}$. Setting $C = \max\{C_f C_{\mathsf{G}},\, L_0,\, L_1\}$ we may infer from Gronwall's lemma that the following holds

$$\begin{aligned}
&\big\|\boldsymbol{a}_\lambda - \boldsymbol{a}^\infty\big\|^2_{C^0([0,T];L^2(\Omega_{\mathrm{act}}))} \\
&\leq Ce^T\Big(\big\|\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty\big\|^2_{L^2(\Omega_T)} + \big\|\mathbf{E}_\lambda - \mathbf{E}_\infty\big\|^2_{L^2(Q_T)} + \big\||\boldsymbol{a}_\lambda - \boldsymbol{a}_\infty|\,|\mathsf{G}^*\mathbf{E}_\infty|\big\|^2_{L^2(\Omega_T)}\Big).
\end{aligned} \tag{4.194}$$

Due to the convergences (4.124) and (4.192), we may infer that the first two summands tend to zero as $\lambda \to \infty$. Involving Lemma A.1.3 and the uniform L^∞-bound on $\boldsymbol{a}_\lambda$ from Lemma 4.3.3 yields that also the third summand tends to zero in the limit. Thus, we have

$$\boldsymbol{a}_\lambda \longrightarrow \boldsymbol{a}^\infty \qquad \text{strongly in } C^0([0,T];L^2(\Omega_{\mathrm{act}})) \qquad \text{as } \lambda \to \infty. \tag{4.195}$$

The uniqueness of the limit (with the usual identification) yields $\boldsymbol{a}_\infty = \boldsymbol{a}^\infty$, thus $\boldsymbol{a}_\infty$ is a solution to (4.70c) with initial condition $\boldsymbol{a}_\infty(x,0) = \boldsymbol{a}_0(x)$ for a.a. $x \in \Omega_{\mathrm{act}}$ in the sense of Definition 4.1.9.

To prove (B), we note that due to the strong convergence (4.195) and the strong convergence $\big(\mathbf{E}_\lambda, \mathbf{H}_\lambda\big) \longrightarrow \big(\mathbf{E}_\infty, \mathbf{H}_\infty\big)$ in $C^0([0,T];L^2(\mathbb{R}^3))$ from (4.192) we can take the limit in the relations

$$\mathsf{P}_\parallel \mathbf{H}_\lambda(t) = 0, \qquad \mathsf{P}_\parallel(\mathbf{E}_\lambda(t) + \mathsf{S}^\lambda \mathsf{G}\boldsymbol{a}_\lambda(t)) = 0, \tag{4.84}$$

uniformly for all $t \in [0,T]$ to get

$$\forall\, t \in [0,T]: \qquad \mathsf{P}_{\|}\mathbf{H}_\infty(t) = 0, \qquad \mathsf{P}_{\|}(\mathbf{E}_\infty(t) + \mathsf{G}\boldsymbol{a}_\infty(t)) = 0. \tag{4.196}$$

This means $\big(\mathbf{E}_\infty, \mathbf{H}_\infty, \boldsymbol{a}_\infty\big) \in C^0([0,T]; \mathbf{L}_{\mathrm{div}})$ and finishes the proof of Theorem 4.2.1.

4.4. Remarks on the Uniqueness Issue

In the articles [JMR00a], [Dum05] and [DuS12] also a uniqueness proof is given. Due to the fact that for given functions $f \in \mathcal{F}_{\mathrm{hyp}}$ and $\mathbf{E} \in L^p(\mathbb{R}^3; \mathbb{R}^3_{\mathbf{E}})$, for all $q \in [1,\infty]$, the function

$$\begin{cases} L^q(\Omega_{\mathrm{act}}) \longrightarrow L^q(\Omega_{\mathrm{act}}) \\ \qquad\quad \boldsymbol{a} \longmapsto f(\cdot, \boldsymbol{a}, \mathsf{G}^*\mathbf{E}) \end{cases} \tag{4.197}$$

is Lipschitz continuous only in the case $p = \infty$, an existence proof for the solutions to Problem 4.1.8 is far from trivial. The proof relies on the following additional assumptions stated in the conjecture below.

Conjecture 4.4.1. *Let the conditions from Theorem 4.2.1 be satisfied and assume that the functions f_0, f_1 are not only continuous, but continuously differentiable w.r.t. $\boldsymbol{a}$. Moreover, assume that the initial data is such that* $\operatorname{curl}\mathbf{E}_0$, $\operatorname{curl}\mathbf{H}_0 \in L^2(\mathbb{R}^3)$. *Then, the solution from Theorem 4.2.1 is unique.*

The uniqueness proofs in [JMR00a], [Dum05] and [DuS12] essentially consist of two parts. In the first step, it is proved that the curl-regularity is propagated, i.e. it holds $\operatorname{curl}\mathbf{E}$, $\operatorname{curl}\mathbf{H} \in C^0([0,T], L^2(\mathbb{R}^3; \mathbb{R}^{3\times 3}_{\mathbf{E}}))$.[28] In the second step, based on estimates for the solenoidal and irrotational parts of the fields involving a Strichartz-type estimate, the actual uniqueness proof is established. Both parts are very technical.

[28] We must warn the reader that in this step, there is a mistake in the proof given in [Dum05]. This mistake is fixed in the proof given in [DuS12]. (Personal communication with É. Dumas.)

A. Appendix

A.1. Basic Results, Miscellaneous

Convergence Results

Theorem A.1.1 (Lebesgue's dominated convergence theorem, [Alt06, Th. 1.23, p. 59])**.** *Let X be a Banach space, $\Omega \subseteq \mathbb{R}^d$ be measurable and $\{f_k\}_{k\in\mathbb{N}}$, $f : \Omega \longrightarrow X$ be measurable functions. Furthermore, consider $\{g_k\}_{k\in\mathbb{N}}$, $g \in L^1(\Omega)$ with $g_k \longrightarrow g$ in $L^1(\Omega)$ for $k \to \infty$. Assume that for $1 \leq p < \infty$ it holds*

$$f_k(x) \longrightarrow f(x) \quad \textit{for almost all } x \in \Omega, \ \textit{for } k \to \infty, \tag{A.1}$$

$$\|f_k(x)\|_X^p \leq |g_k(x)| \quad \textit{for almost all } x \in \Omega, \ \textit{for all } k \in \mathbb{N}. \tag{A.2}$$

Then, it holds $\{f_k\}_{k\in\mathbb{N}}$, $f \in L^p(\Omega; X)$ and $f_k \longrightarrow f$ in $L^p(\Omega; X)$ for $k \to \infty$.

Theorem A.1.2 (H. Weyl 1909, [Els05, Corollary VI.2.7, p. 232])**.** *Let X be a Banach space, $\Omega \subseteq \mathbb{R}^d$ be measurable, $1 \leq p \leq \infty$ and $\{f_k\}_{k\in\mathbb{N}}$, $f \in L^p(\Omega; X)$. If it holds*

$$\|f_k - f\|_{L^p(\Omega;X)} \longrightarrow 0 \qquad \textit{for } k \to \infty, \tag{A.3}$$

then, there exists a subsequence $\{f_{k_j}\}_{j\in\mathbb{N}} \subset \{f_k\}_{k\in\mathbb{N}}$ that satisfies

$$f_{k_j}(x) \longrightarrow f(x) \qquad \textit{pointwise for a.a. } x \in \Omega \textit{ as } j \to \infty. \tag{A.4}$$

Lemma A.1.3. *Let $\Omega \subseteq \mathbb{R}^d$ be measurable and let $p, q, s \in [1, \infty]$ with $p^{-1} + q^{-1} = s^{-1}$ and some $1 \leq r \leq p$ be given. Consider functions g, f, $\{f_k\}_{k\in\mathbb{N}} \subset L^p(\Omega)$ satisfying*

$$f_k \longrightarrow f \quad \textit{strongly in } L^r(\Omega) \quad \textit{for } k \to \infty \tag{A.5}$$

$$|f_k(x)| \leq |g(x)| \quad \textit{for a.a. } x \in \Omega. \tag{A.6}$$

Then, for every function $h \in L^q(\Omega)$, the following convergence holds

$$\int_\Omega \big||f_k(x) - f(x)|\,|h(x)|\big|^s \, dx \longrightarrow 0 \qquad \textit{for } k \to \infty. \tag{A.7}$$

Proof. We define the series of functions

$$b_k(x) := \big||f_k(x) - f(x)|\,|h(x)|\big|^s \in L^1(\Omega) \tag{A.8}$$

and the series of positive real numbers

$$B_k := \int_\Omega \big| |f_k(x) - f(x)|\,|h(x)| \big|^s \, dx. \tag{A.9}$$

The set $\{B_k\}_{k\in\mathbb{N}} \subset [0,\infty)$ is bounded since Hölder's inequality[1] yields

$$B_k \le \big(\|f_k\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} \big)^s \, \|h\|^s_{L^q(\Omega)} \le \big(\|g\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} \big)^s \, \|h\|^s_{L^q(\Omega)}. \tag{A.10}$$

Therefore, we may infer the existence of some $B_\infty \ge 0$ and some subsequence $\{B_{k_j}\}_{j\in\mathbb{N}}$ with

$$B_\infty := \limsup_{k\to\infty} B_k = \lim_{j\to\infty} B_{k_j}. \tag{A.11}$$

Due to the convergence in (A.5), we can infer from Weyl's theorem that there exists a further subsequence of $\{b_{k_j}\}_{j\in\mathbb{N}}$, say $\{b_{k_{j_l}}\}_{l\in\mathbb{N}}$ such that $b_{k_{j_l}}(x) \longrightarrow 0$ pointwise for a.a. $x \in \Omega$ as $l \to \infty$. Furthermore, the function $\widetilde{g}(x) := \big(|g(x)| + |f(x)|\big)^s \, |h(x)|^s \in L^1(\Omega)$ is a majorant to $\{b_k\}_{k\in\mathbb{N}}$. Thus, Lebesgue's theorem yields the convergence

$$\int_\Omega b_{k_{j_l}}(x)\, dx \longrightarrow 0 \quad \text{for } l \to \infty. \tag{A.12}$$

This implies $B_\infty = 0$. Obviously, we also have $0 \le \lim_{k\to\infty} B_k \le \limsup_{k\to\infty} B_k = 0$. This shows (A.7). $\square$

The following simple result was not to be found in the literature.

Lemma A.1.4. *Let $\big(H, (\cdot,\cdot)_H\big)$ be a Hilbert space and let linear operators A, $\{\mathsf{A}_k\}_{k\in\mathbb{N}} \subset \mathcal{L}(H,H)$ with adjoints A^*, $\{\mathsf{A}_k^*\}_{k\in\mathbb{N}}$ be given such that*

$$\forall\, u \in H: \qquad \big\| \mathsf{A}_k^* u - \mathsf{A}^* u \big\|_H \longrightarrow 0. \tag{A.13}$$

Moreover, let u, $\{u_k\}_{k\in\mathbb{N}} \subset H$ be given such that

$$u_k \longrightarrow u \qquad \textit{weakly in } H. \tag{A.14}$$

Then, it holds

$$\mathsf{A}_k u_k \longrightarrow \mathsf{A} u \qquad \textit{weakly in } H. \tag{A.15}$$

Proof. It holds

$$\begin{aligned}
&\forall v \in H: \qquad \big| \big(\mathsf{A}_k\, u_k - \mathsf{A}\, u, v\big)_H \big| = \big| \big((\mathsf{A}_k - \mathsf{A}) u_k, v\big)_H + \big(\mathsf{A}(u_k - u), v\big)_H \big| \\
&= \big| \big(u_k, (\mathsf{A}_k^* - \mathsf{A}^*) v\big)_H + \big((u_k - u), \mathsf{A}^* v\big)_H \big| \\
&\le \|u_k\|_H \big\| (\mathsf{A}_k^* - \mathsf{A}^*) v \big\|_H + \big| \big((u_k - u), \mathsf{A}^* v\big)_H \big|.
\end{aligned} \tag{A.16}$$

[1] See [Alt06, Lemma 1.16, p. 51]

Due to the boundedness of $\|u_k\|_H$ and the convergence (A.13) we get that the first summand tends to zero. Due to the weak convergence (A.14), also the second summand tends to zero. This proves the claim. $\square$

Next, we state a rather general variant of the well-known Arzelà-Ascoli theorem. First, we give the definition of *equi-continuity* in this context.

Definition A.1.5 (equi-continuity, [Dug66, p. 266])**.** *Let (Y, d) be a metric space and let X be an arbitrary (topological) space. A subset $\mathrm{A} \subset C^0(X, Y)$ is called equi-continuous at $x_0 \in X$ if*

$$\forall\, \epsilon > 0, \quad \exists\, U(x_0) \subset X \quad \text{such that} \quad \forall\, f \in \mathrm{A} \text{ it holds } f(U(x_0)) \subset B_\epsilon(f(x_0)). \tag{A.17}$$

We say that A is equi-continuous on X whenever A is equi-continuous for every $x \in X$.

Theorem A.1.6 (Arzelà-Ascoli, [Dug66, p. 267])**.** *Let (Y, d) be a metric space and let X be an arbitrary (topological) space. Assume that $\mathrm{A} \subset C^0(\mathrm{X}, \mathrm{Y})$ has the following properties.*

(i) The set A is equi-continuous on X.

(ii) For every $x \in \mathrm{X}$ the set $\overline{\{f(x) : f \in \mathrm{A}\}} \subset Y$ is compact.

Then, the closure $\overline{\mathrm{A}}$ of A is compact in $C^0(\mathrm{X}, \mathrm{Y})$ and equi-continuous on X.

Our main interest lies in the following corollary.

Corollary A.1.7. *For an arbitrary $d \in \mathbb{N}$ let $\Omega \subseteq \mathbb{R}^d$ be open and let $T > 0$ be given. We consider a given bounded set $\mathrm{A} \subset W^{1,\infty}([0, T], L^2(\Omega))$. There exists a function $a \in \overline{\mathrm{A}}$ and sequence $\{a_k\}_{k\in\mathbb{N}} \subset \mathrm{A}$ such that*

$$\forall\, t \in [0, T] : \qquad a_k(t) \longrightarrow a(t) \qquad \text{weakly in } L^2(\Omega). \tag{A.18}$$

The proof of this corollary is based on the following observation concerning the metrizability of weak topologies.

Theorem A.1.8 ([DuS57, Theorem 3, p. 434])**.** *The weak topology of a weakly compact subspace A of a separable Banach space X is metrizable. In particular, this means that there exists a metric d such that for all $x_0 \in \overline{\mathrm{A}}$ and for all $\{x_k\}_{k\in\mathbb{N}} \subset \mathrm{A}$ we have*

$$d(x_k, x_0) \longrightarrow 0 \qquad \Longleftrightarrow \qquad \forall\, f \in \mathrm{X}^* : \ f(x_k - x_0) \longrightarrow 0. \tag{A.19}$$

Proof of Corollary A.1.7. Due to the boundedness of A and the continuity of the embedding[2] $W^{1,\infty}([0, T]; L^2(\Omega)) \hookrightarrow C^0([0, T]; L^2(\Omega))$ we may infer the existence of a constant C_I such that

$$\sup_{k\in\mathbb{N}} \max_{t\in[0,T]} \|a_k(t)\|_{L^2(\Omega)} \le C_I. \tag{A.20}$$

[2]See [Zei90, Problem 23.13a, p. 450].

In particular, denoting with τ_{weak} the weak topology of $L^2(\Omega)$, this implies that we have[3]

$$\forall\, t \in [0,T]: \qquad \overline{\{a(t) : a \in \mathrm{A},\, t \in [0,T]\}} \subset L^2(\Omega) \qquad \text{is compact in } (L^2(\Omega), \tau_{\text{weak}}).$$

From Theorem A.1.8 we may infer that

$$\text{the topological space} \quad \Big(\overline{\{a(t) : a \in \mathrm{A},\, t \in [0,T]\}},\, \tau_{\text{weak}} \Big) \quad \text{is metrizable.}$$

We denote this metric space with (Y,d) for a moment. The boundedness of A implies that A is uniformly Lipschitz continuous on $[0,T]$ with image in both, the normed space $L^2(\Omega)$ and the metric space (Y,d), since the strong topology is finer than the weak one. Thus, Theorem A.1.6 yields that $\overline{\mathrm{A}} \subset C^0([0,T],Y)$ is compact. In particular, this implies the existence of some sequence $\{a_k\}_{k\in\mathbb{N}} \subset \mathrm{A}$ and some function $a \in \overline{\mathrm{A}}$ with

$$\max_{t\in[0,T]} d(a_k(t), a(t)) \longrightarrow 0, \qquad k \to \infty. \tag{A.21}$$

In view of Theorem A.1.8 this is equivalent to

$$\forall\, u \in L^2(\Omega) \quad \text{it holds} \quad \max_{t\in[0,T]} \int_\Omega u\,\big(a_k(t) - a(t)\big)\, dx \longrightarrow 0, \quad k \to \infty. \tag{A.22}$$

This was the assertion. $\square$

Estimates

Theorem A.1.9 (Gronwall's lemma)**.** *Let $T > 0$ and $p,\, q \in [1,\infty]$ be conjugated exponents (i.e. $p^{-1} + q^{-1} = 1$) and let non-negative functions $u \in L^p((0,T))$, $h \in L^q((0,T))$ and $g \in C^0([0,T])$ be given. If for a.e. $t \in (0,T)$ the estimate*

$$u(t) \le g(t) + \int_0^t h(s)\, u(s)\, ds \tag{A.23}$$

is satisfied, then, the following holds.

(i) With $H(t) := \int_0^t h(s)\, ds$ we have for a.e. $t \in (0,T)$,

$$u(t) \le g(t) + e^{H(t)} \int_0^t e^{-H(s)} h(s)\, g(s)\, ds. \tag{A.24}$$

(ii) If moreover, g is non-decreasing, we have for a.e. $t \in (0,T)$,

$$u(t) \le g(t)\, \exp\Big(\int_0^t h(s)\, ds \Big). \tag{A.25}$$

[3] See [Wer11, p. 417].

Proof. For (i), we use the proof from [Rau12, p. 50]. We denote by γ the absolutely continuous function

$$\gamma(t) := \int_0^t h(s)\, u(s)\, ds. \tag{A.26}$$

Then, due to (A.23) we have for a.e. $t \in (0, T)$

$$\gamma'(t) = h(t)\, u(t) \leq h(t)\, g(t) + h(t)\, \gamma(t). \tag{A.27}$$

Therefore, it holds

$$\Big(e^{-H(t)}\gamma(t)\Big)' = e^{-H(t)}\Big(\gamma'(t) - h(t)\,\gamma(t)\Big) \leq e^{-H(t)}h(t)\, g(t). \tag{A.28}$$

Since $\gamma(0) = 0$, integrating this inequality over $(0, t)$ for an arbitrary $t \in [0, T]$ yields

$$e^{-H(t)}\gamma(t) \leq \int_0^t e^{-H(s)}h(s)\, g(s)\, ds. \tag{A.29}$$

Inserting this into (A.23) yields (A.24).
To prove (ii), note that it holds

$$-\frac{d}{dt}\exp\Big(-\int_0^t h(s)\, ds\Big) = h(t)\, \exp\Big(-\int_0^t h(s)\, ds\Big). \tag{A.30}$$

In the case that g is non-decreasing, we get the following estimate for a.e. $t \in (0, T)$

$$u(t) \leq g(t) + e^{H(t)}\int_0^t e^{-H(s)}h(s)\, g(s)\, ds \leq g(t) + g(t)\, e^{H(t)}\int_0^t e^{-H(s)}h(s)\, ds. \tag{A.31}$$

Insertion of (A.30) yields

$$\begin{aligned} u(t) &\leq g(t) - g(t)\, e^{H(t)}\int_0^t \frac{d}{ds}\exp\Big(-\int_0^s h(\tau)\, d\tau\Big)\, ds \\ &\leq g(t) - g(t)\, e^{H(t)}\Big[e^{-H(t)} - 1\Big] \\ &\leq g(t) - g(t)\Big(1 - e^{H(t)}\Big) \leq g(t)\, \exp\Big(\int_0^t h(s)\, ds\Big). \end{aligned} \tag{A.32}$$

This was the claim. □

Lemma A.1.10 (Convolution Estimate, [Alt06, p. 107])**.** *Let $p \in [1, \infty]$, $f \in L^p(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d)$. Then,*

$$(f * g)(x) := \int_{\mathbb{R}^d} f(x - y)\, g(y)\, dy \tag{A.33}$$

defines a function $f * g \in L^p(\mathbb{R}^d)$ called the convolution of f and g. Moreover, it holds $f * g = g * f$ and we have the following convolution estimate

$$\|f * g\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)}\, \|g\|_{L^1(\mathbb{R}^d)}. \tag{A.34}$$

For bounded domains $\Omega \subset \mathbb{R}^d$ the inclusion $L^q(\Omega) \subset L^q(\Omega)$ if $1 \leq q \leq p \leq \infty$ is well known. For the whole space this inclusion does not hold, but we have

Lemma A.1.11 (Interpolation Estimate, [AdF03, Theorem 2.11, p. 27])**.** *Let $\Omega \subseteq \mathbb{R}^d$ be open and let $1 \leq p < q < r \leq \infty$ be given so that for some $\vartheta \in (0,1)$ it holds*

$$\frac{1}{q} = \frac{\vartheta}{p} + \frac{1-\vartheta}{r}. \tag{A.35}$$

If $u \in L^p(\Omega) \cap L^r(\Omega)$, then $u \in L^q(\Omega)$ and the following estimate holds.

$$\|u\|_{L^q(\Omega)} \leq \|u\|_{L^p(\Omega)}^{\vartheta} \|u\|_{L^r(\Omega)}^{1-\vartheta}. \tag{A.36}$$

Lemma A.1.12 ([Sta05, Proposition 5.8])**.** *There exists a constant $C_\parallel > 0$ such that for all $p \in (2,\infty)$, the operator $\mathsf{P}_\parallel$ is bounded as an operator mapping $L^p(\mathbb{R}^d)$ into itself with norm less than $C_\parallel\, p$.*

Lemma A.1.13. *Let $\mathbb{T}$ be a placeholder for $\mathbb{R}$ or $\mathbb{S}^1$, let $\alpha \in \mathbb{R}$ and let functions w, $\{w_k\}_{k\in\mathbb{N}} \subset L^\infty((0,T);(L^2 \cap L^\infty)(\mathbb{T}))$ be given with*

$$\sup_{k\in\mathbb{N}} \|w_k\|_{L^\infty((0,T)\times\mathbb{T})} \leq \|w\|_{L^\infty((0,T)\times\mathbb{T})}. \tag{A.37}$$

We set $C_\alpha(w,T) := |\alpha| \exp(|\alpha|\, T\, \|w\|_{L^\infty((0,T)\times\mathbb{T})})$. Then, for all $t \in [0,T]$ and for a.a. $x \in \mathbb{T}$ we have

$$\begin{aligned}
&\Big|\exp\Big(-\alpha\int_0^t w(x+\tau-t,\tau)\,d\tau\Big) - \exp\Big(-\alpha\int_0^t w_k(x+\tau-t,\tau)\,d\tau\Big)\Big| \\
&\leq C_\alpha(w,T)\int_0^t \big|w(x+\tau-t,\tau) - w_k(x+\tau-t,\tau)\big|\,d\tau.
\end{aligned} \tag{A.38}$$

In particular, for all $p \in [2,\infty]$ and for all $t \in [0,T]$, the following estimate holds true.

$$\begin{aligned}
&\Big\|\exp\Big(-\alpha\int_0^t w(x+\tau-t,\tau)\,d\tau\Big) - \exp\Big(-\alpha\int_0^t w_k(x+\tau-t,\tau)\,d\tau\Big)\Big\|_{L^p(\mathbb{T})} \\
&\leq C_\alpha(w,T)\int_0^t \big\|w(\tau) - w_k(\tau)\big\|_{L^p(\mathbb{T})}\,d\tau.
\end{aligned} \tag{A.39}$$

Proof. Note that on a bounded interval, say $I = [a,b]$, the function $x \mapsto \exp(|\alpha|\,x)$ is Lipschitz continuous with Lipschitz constant $L = |\alpha|\,\exp(|\alpha|\,b)$. Therefore, for some given function $w \in L^\infty((0,T)\times\mathbb{T})$, for a.e. $x \in \mathbb{T}$ the function

$$\int_0^t w(x+\tau-t,\tau)\,d\tau \longmapsto \exp\Big(-\alpha\int_0^t w(x+\tau-t,\tau)\,d\tau\Big)$$

is Lipschitz continuous with Lipschitz constant $L_{\exp} \leq |\alpha|\exp(|\alpha|\,t\,\|w\|_{L^\infty((0,T)\times\mathbb{T})}) \leq |\alpha|\exp(|\alpha|\,T\,\|w\|_{L^\infty((0,T)\times\mathbb{T})}) =: C_\alpha(w,T)$. Due to the uniform bounds in (A.37), this implies (A.38) for all $t \in [0,T]$ and for a.a. $x \in \mathbb{T}$. Furthermore, considering the L^p-norm, the above Lipschitz continuity implies the following estimate

$$\begin{aligned}
&\Big\| \exp\Big(-\alpha \int_0^t w(x+\tau-t,\tau)\,d\tau\Big) - \exp\Big(-\alpha \int_0^t w_k(x+\tau-t,\tau)\,d\tau\Big)\Big\|_{L^p(\mathbb{T})} \\
&\quad \le C_\alpha(w,T) \Big\| \int_0^t w(x+\tau-t,\tau) - w_k(x+\tau-t,\tau)\,d\tau \Big\|_{L^p(\mathbb{T})} \\
&\quad \le C_\alpha(w,T) \int_0^t \big\| w(x+\tau-t,\tau) - w_k(x+\tau-t,\tau)\big\|_{L^p(\mathbb{T})}\,d\tau \\
&\quad = C_\alpha(w,T) \int_0^t \big\| w(\tau) - w_k(\tau)\big\|_{L^p(\mathbb{T})}\,d\tau.
\end{aligned}$$

This proves Lemma A.1.13 □

Function Spaces

The set in the following definition is often called the set of test functions.

Definition A.1.14. *Let $d \in \mathbb{N}$ and $\Omega \subseteq \mathbb{R}^d$ open. We define the set $C_c^\infty(\Omega)$ as the subset of functions φ from $C^\infty(\Omega)$ such that $\operatorname{supp}(\varphi) \subset \Omega$ is compact.*

We give the following definition of Sobolev spaces[4] and note that there is an equivalent characterization, see Definition A.3.5.

Definition A.1.15. *Let $d \in \mathbb{N}$ and $\Omega \subseteq \mathbb{R}^d$ open. For $m \in \mathbb{N}_0$ and $p \in [1,\infty]$, we define the Sobolev space $W^{m,p}(\Omega)$ as the subset of functions u from $L^p(\Omega)$ such that for every multi-index $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le m$, there exists a function $u^{(\alpha)} \in L^p(\Omega)$ such that for all $\varphi \in C_c^\infty(\Omega)$ it holds*

$$\int_\Omega \partial_x^\alpha \varphi(x) \cdot u\,dx = (-1)^{|\alpha|} \int_\Omega \varphi \cdot u^{(\alpha)}\,dx. \tag{A.40}$$

Moreover, we define the corresponding Sobolev norms $\|\cdot\|_{W^{m,p}(\Omega)}$ by

$$\|u\|_{W^{m,p}(\Omega)} := \sum_{|\alpha|\le m} \|u^{(\alpha)}\|_{L^p(\Omega)}. \tag{A.41}$$

For $p < \infty$ we define $W_0^{m,p}(\Omega) := \overline{C_c^\infty(\Omega) \cap W^{m,p}(\Omega)}^{\|\cdot\|_{W^{m,p}(\Omega)}}$ and set $H^m(\Omega) := W^{m,2}(\Omega)$.

The following definition or characterization of the duals of the Sobolev spaces $H_0^m(\Omega)$ where $H_0^m(\Omega) := W_0^{m,2}(\Omega)$ can for example be found in [AdF03, 3.13, p. 64–65].

[4] See also [Alt06, 1.25, p. 62–65].

Definition A.1.16. *Let $\Omega \subseteq \mathbb{R}^d$ denote an arbitrary open set and let $m \in \mathbb{N}$. We consider the space $H_0^m(\Omega)$. Often, the dual space of $H_0^m(\Omega)$ denoted with $\big(H_0^m(\Omega)\big)^*$ is introduced as the completion of $L^2(\Omega)$ with respect to the norm*

$$\|u\|_{H^{-m}(\Omega)} := \sup_{\substack{\varphi \in H_0^m(\Omega) \\ \|\varphi\|_{H_0^m(\Omega)}=1}} \Big| \int_\Omega u \cdot \varphi \, dx \, ds \Big|. \tag{A.42}$$

Next, we introduce the definition of equi-integrability.

Definition A.1.17 (equi-integrable, [ABM06, Def. 2.4.4])**.** *Let $\Omega \subset \mathbb{R}^d$ be bounded and let* A *be a subset of $L^1(\Omega)$. We say that* A *is equi-integrable if the following two conditions hold:*

(i) A *is bounded in $L^1(\Omega)$,*

(ii) for every $\epsilon > 0$ there exists some $\delta(\epsilon) > 0$ such that for all measurable sets $M \subseteq \Omega$, it holds

$$\mu(M) < \delta(\epsilon) \Longrightarrow \sup_{f \in \mathrm{A}} \int_M |f(x)| \, dx < \epsilon.$$

We introduce a criterion for the equi-integrability of a given set of L^1-functions. The cited result is stated in an analytical manner. Another very popular reference for the same result is [Mey66].

Theorem A.1.18 (De La Vallée–Poussin Theorem, [ABM06, Th. 2.4.4])**.** *Let $\Omega \subset \mathbb{R}^d$ be bounded and let* A *be a subset of $L^1(\Omega)$. Then, the following properties are equivalent:*

(i) A *is equi-integrable,*

(ii) there exists a function $\theta : [0, \infty) \longrightarrow [0, \infty)$ such that $\lim_{s \to \infty} \frac{\theta(s)}{s} = \infty$ and

$$\sup_{f \in \mathrm{A}} \int_\Omega \theta(|f(x)|) \, dx < \infty.$$

The function θ can be taken convex and increasing.

In particular, this implies that for all $\sigma > 0$, any bounded set $\mathrm{A} \subset L^{1+\sigma}(\Omega)$ is equi-integrable.

Miscellaneous

Lemma A.1.19. *For all matrices A, $B \in \mathbb{C}^{N \times N}$ with $N \in \mathbb{N}$ it holds*

$$\mathrm{Tr}\big(A[A, B]\big) = 0 \qquad \text{and} \qquad \mathrm{Tr}\big([A, B]\big) = 0, \tag{A.43}$$

where $[A, B] = AB - BA$ is the commutator of the matrices.

Proof. We have

$$\operatorname{Tr}\big([A,B]\big) = \sum_{j,k=1}^{N} \big(a_{jk}b_{kj} - a_{kj}b_{jk}\big) = \sum_{j,k=1}^{N} a_{jk}b_{kj} - \sum_{j,k=1}^{N} a_{kj}b_{jk} = \sum_{j,k=1}^{N} a_{jk}b_{kj} - \sum_{j,k=1}^{N} a_{jk}b_{kj}$$

and

$$\begin{aligned}\operatorname{Tr}\big(A[A,B]\big) &= \sum_{j=1}^{N}(A(AB))_{jj} - (A(BA))_{jj} = \sum_{j=1}^{N}\Big(\sum_{k=1}^{N} a_{jk}\sum_{l=1}^{N}\big(a_{kl}b_{lj} - b_{kl}a_{lj}\big)\Big)\\ &= \sum_{j,k,l=1}^{N} a_{jk}\big(a_{kl}b_{lj} - b_{kl}a_{lj}\big) = \sum_{j,k,l=1}^{N} a_{jk}a_{kl}b_{lj} - a_{jk}a_{lj}b_{kl}.\end{aligned}$$

We sum over all triples $(j,k,l) \in \{1,\cdots,N\}^3$. For every triple (j,k,l) there exists exactly one other triple $(\hat{j},\hat{k},\hat{l})$ with $\hat{j}=k$, $\hat{k}=l$, $\hat{l}=j$, so that we get

$$a_{jk}a_{kl}b_{lj} - a_{\hat{j}\hat{k}}a_{\hat{l}\hat{j}}b_{\hat{k}\hat{l}} = a_{jk}a_{kl}b_{lj} - a_{jk}a_{kl}b_{lj} = 0.$$

We also have for every triple (j,k,l) some other triple $(\hat{j},\hat{k},\hat{l})$ with $\hat{l}=k$, $\hat{j}=l$, $\hat{k}=j$, so that we get

$$a_{\hat{j}\hat{k}}a_{\hat{k}\hat{l}}b_{\hat{l}\hat{j}} - a_{jk}a_{lj}b_{kl} = a_{jk}a_{lj}b_{kl} - a_{jk}a_{lj}b_{kl} = 0.$$

That concludes the proof. □

Lemma A.1.20. *For all $\alpha \geq 2$, the function $h_\alpha(x) := x\cdot\big(\log(1+x)-\log(1-x)\big) - |x|^\alpha$ satisfies $h_\alpha(x) \geq 0$ for all $x \in [-1,1]$. The statement is false for all $\alpha < 2$.*

Proof. We restrict our considerations to non-negative x, i.e. $x \in [0,1]$. In this case, the first two derivatives of h_α are given by

$$h'_\alpha(x) = \frac{x}{1+x} + \frac{x}{1-x} + \log\Big(\frac{1+x}{1-x}\Big) - \alpha x^{\alpha-1}, \tag{A.44}$$

$$h''_\alpha(x) = \frac{2+x}{(1+x)^2} + \frac{2-x}{(1-x)^2} + -\alpha(\alpha-1)x^{\alpha-2}. \tag{A.45}$$

Obviously, it needs to hold $h_\alpha(0) \geq 0$. This implies that α has to be strictly positive. Moreover, in the case $h_\alpha(0)=0$ it must hold $h'_\alpha(0)=0$. This implies that α must not be equal to 1. Furthermore, in a neighborhood of 0, the function h_α has to be convex. We have

$$h''_\alpha(x) \geq 0 \quad\Longleftrightarrow\quad (\alpha^2-\alpha)x^{\alpha-2} \leq \frac{4}{(1-x^2)^2}. \tag{A.46}$$

Since for all $0 < \alpha < 2$ we have $\lim_{x\searrow 0} x^{\alpha-2} = \infty$, the exponent α must satisfy $\alpha \geq 2$. Moreover, the above equation implies that for all $\alpha \geq 2$, the function h_α is convex. Clearly, for all $\alpha \geq 2$, the function h_α satisfies $h'_\alpha(0)=0$ and $h_\alpha(0)=0$. Thus, for all $\alpha \geq 2$ we have $h_\alpha \geq 0$. □

A.2. Bloch Coordinates

The space of $\mathbb{C}^{2\times 2}_{\text{herm}}$ is isomorphic to the space $\mathbb{R}^4$. For example, the unit matrix together with the Pauli matrices

$$A_0 = \mathsf{Id}_{2\times 2}, \quad A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{A.47}$$

form a real basis of the space $\mathbb{C}^{2\times 2}_{\text{herm}}$ and the mapping

$$\tau : \mathbb{R}^4 \longrightarrow \mathbb{C}^{2\times 2}_{\text{herm}}, \qquad \tau : (a_0, \boldsymbol{a}) \longmapsto a_0 A_0 + \sum_{j=1}^{3} a_j A_j \tag{A.48}$$

is a diffeomorphism. We call the coordinates $(a_0, \boldsymbol{a})$ *Bloch coordinates of the matrix* $\Omega = \tau(a_0, \boldsymbol{a})$. On the other hand, any given Hermitian matrix $\Omega \in \mathbb{C}^{2\times 2}_{\text{herm}}$ with the entries $(\omega_{jk})_{j,k=1,2}$ can be identified by its Bloch coordinates $(a_0, \boldsymbol{a})$, via the inverse of τ given by

$$\tau^{-1} : \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \longmapsto \begin{pmatrix} (\omega_{11} + \omega_{22})/2 \\ (\omega_{11} - \omega_{22})/2 \\ (\omega_{12} + \omega_{21})/2 \\ (\omega_{12} - \omega_{21})/(2i) \end{pmatrix} \tag{A.49}$$

by setting $(a_0, \boldsymbol{a})^T := \tau^{-1}(\Omega)$. Note, that for all $\Omega \in \mathbb{C}^{2\times 2}_{\text{herm}}$ we have $\tau\big(\tau^{-1}(\Omega)\big) = \Omega$ and that $(\tau^{-1})^* = \frac{1}{2}\tau$.

In Bloch coordinates, we have the following rules to calculate traces, commutators, matrix products and eigenvalues.

Lemma A.2.1. *Let matrices A, B, $C \in \mathbb{C}^{2\times 2}_{herm}$ be given such that $A = \tau(a_0, \boldsymbol{a})$, $B = \tau(b_0, \boldsymbol{b})$ and $C = \tau(c_0, \boldsymbol{c})$. Then, it holds*

(i) $\operatorname{Tr}\big(AB\big) = 2\big(a_0 b_0 + \boldsymbol{a} \cdot \boldsymbol{b}\big)$.

(ii) $\big[A, B\big] = -i\,\tau\big(0,\, 2\boldsymbol{a} \times \boldsymbol{b}\big) = \tau\big(0,\, -2i(\boldsymbol{a} \times \boldsymbol{b})\big)$.

(iii) $AB = \tau\big((a_0 b_0 + \boldsymbol{a} \cdot \boldsymbol{b}),\, (a_0\boldsymbol{b} + b_0\boldsymbol{a}) - i\boldsymbol{a} \times \boldsymbol{b}\big)$.

(iv) $$ABC = \tau\Big(\big((a_0 b_0 + \boldsymbol{a} \cdot \boldsymbol{b})c_0 + (a_0\boldsymbol{b} + b_0\boldsymbol{a} - i\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}\big),$$
$$\big((a_0 b_0 + \boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} + (a_0\boldsymbol{b} + b_0\boldsymbol{a} - i\boldsymbol{a} \times \boldsymbol{b})c_0 - i(a_0\boldsymbol{b} + b_0\boldsymbol{b} - i\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}\big)\Big).$$

(v) The eigenvalues of A are given by $a_+ = (a_0 + |\boldsymbol{a}|)$ and $a_- = (a_0 - |\boldsymbol{a}|)$.

We emphasize that (i) yields $\operatorname{Tr}(A) = 2a_0$ and that we have

$$\tau^{-1}\big(\mathcal{R}_2\big) = \big\{(a_0, \boldsymbol{a}) \in \mathbb{R}^4 : a_0 = 1/2,\ |\boldsymbol{a}| \leq 1/2\big\}. \tag{A.50}$$

Thus, defining the *Bloch ball*

$$\mathcal{A} := \big\{ \boldsymbol{a} = (a_1, a_2, a_3) \in \mathbb{R}^3_{\boldsymbol{a}} :\ |\boldsymbol{a}| \leq 1/2 \big\}, \tag{A.51}$$

we can identify $\mathcal{R}_2$ with $\mathcal{A}$ via

$$\mathcal{R}_2 \ni \rho = \widehat{\rho}(\boldsymbol{a}) = \tau(1/2, \boldsymbol{a}) = \begin{pmatrix} \frac{1}{2} + a_1 & a_2 + ia_3 \\ a_2 - ia_3 & \frac{1}{2} - a_1 \end{pmatrix}. \tag{A.52}$$

We call the elements of $\mathcal{A}$ *Bloch vectors* and for a given matrix $\rho = \widehat{\rho}(\boldsymbol{a}) \in \mathcal{R}_2$ we call $\boldsymbol{a}$ the *Bloch vector of the matrix* ρ. The sphere of the Bloch ball $\mathcal{A}$ consists of the pure states[5]. Moreover, for a continuous function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ and a Hermitian matrix H we can define the Hermitian matrix $\phi(H)$ by replacing the eigenvalues λ_j with $\phi(\lambda_j)$ in the spectral decomposition. In Bloch coordinates the following is analoguos.

Definition A.2.2. *Let $(a_0, \boldsymbol{a}) \in \mathbb{R}^4$ and $\phi \in C^0(\mathbb{R}, \mathbb{R})$ be given. For the Hermitian matrix $A := \tau(a_0, \boldsymbol{a})$ we define $\phi(A)$ to be*

$$\phi(A) := \tau\big(\Phi_0(a_0, \boldsymbol{a}), \Phi_1(a_0, \boldsymbol{a})\boldsymbol{a}\big) \tag{A.53a}$$

with

$$\Phi_0(a_0, \boldsymbol{a}) := \frac{1}{2}\big(\phi(a_0 + |\boldsymbol{a}|) + \phi(a_0 - |\boldsymbol{a}|)\big) \quad \textit{and} \tag{A.53b}$$

$$\Phi_1(a_0, \boldsymbol{a}) := \frac{1}{2|\boldsymbol{a}|}\Big(\phi(a_0 + |\boldsymbol{a}|) - \phi(a_0 - |\boldsymbol{a}|)\Big). \tag{A.53c}$$

Examples

1. For $A = \tau(a_0, \mathbf{a})$ and $\phi(A) = \log(A)$, we have

$$\log(A) = \tau\Big(\frac{1}{2}\log\big(a_0^2 - |\boldsymbol{a}|^2\big),\ \frac{1}{2|\boldsymbol{a}|}\big(\log(a_0 + |\boldsymbol{a}|) - \log(a_0 - |\boldsymbol{a}|)\big)\boldsymbol{a}\Big). \tag{A.54}$$

   In particular, for $\rho = \widehat{\rho}(\boldsymbol{a})$, we have

$$\log\big(\widehat{\rho}(\boldsymbol{a})\big) = \tau\Big(\frac{1}{2}\log\big(\tfrac{1}{4} - |\boldsymbol{a}|^2\big),\ \frac{\boldsymbol{a}}{\lambda(|\boldsymbol{a}|)}\Big) \tag{A.55}$$

   where $\lambda(|\boldsymbol{a}|)$ is defined[6] for $\boldsymbol{a} \in \mathcal{A}$ by

$$\lambda(|\boldsymbol{a}|) := \frac{2|\boldsymbol{a}|}{\log(1/2 + |\boldsymbol{a}|) - \log(1/2 - |\boldsymbol{a}|)}. \tag{A.56}$$

[5] Often people mistakenly think that only the poles (i.e. $a_1 = \pm 1/2$) of the Bloch ball correspond to pure states. We stress that this is wrong. The poles of the Bloch ball are the pure states of the basis elements (usually the eigenfunctions of the unperturbed Hamiltonian).

[6] In fact, λ is well defined for all $\boldsymbol{a} \in \{\boldsymbol{a} \in \mathcal{A} :\ |\boldsymbol{a}| < 1/2\}$, but can be continuously extended to $\mathcal{A}$.

2. If $\rho \in \mathcal{R}_2$ has the representation $\rho = \tau(1/2, \boldsymbol{a})$ and $B \in \mathbb{C}^{2\times 2}_{\mathrm{herm}}$ has the representation $B = \tau(b_0, \boldsymbol{b})$, we have

$$\mathcal{C}_\rho B = \tau\Big(\frac{1}{2}b_0 + \boldsymbol{a}\cdot\boldsymbol{b},\ \frac{b_0}{2}\boldsymbol{a} + \mathsf{C}_{\boldsymbol{a}}\boldsymbol{b}\Big), \qquad \mathsf{C}_{\boldsymbol{a}} := \lambda(|\boldsymbol{a}|)\,\mathsf{Id}_{3\times 3} + \mu(|\boldsymbol{a}|)\,\boldsymbol{a}\otimes\boldsymbol{a} \quad \text{(A.57)}$$

where the functions λ and μ are defined[7] for $\boldsymbol{a} \in \mathcal{A}$ by

$$\lambda(|\boldsymbol{a}|) := \frac{2|\boldsymbol{a}|}{\log(1/2 + |\boldsymbol{a}|) - \log(1/2 - |\boldsymbol{a}|)}, \qquad \mu(|\boldsymbol{a}|) := \frac{1 - 2\lambda(|\boldsymbol{a}|)}{2|\boldsymbol{a}|^2}. \quad \text{(A.58)}$$

To see this, write the integrand of the canonical correlation operator given by $\widehat{\rho}(\boldsymbol{a})^{1-s}\,\tau(b_0,\boldsymbol{b})\,\widehat{\rho}(\boldsymbol{a})^s$ according to Lemma A.2.1 (iv) and Definition A.2.2. We recall that the eigenvalues of $\widehat{\rho}(\boldsymbol{a})$ are given by $a_+ = (1/2 + |\boldsymbol{a}|)$ and $a_- = (1/2 - |\boldsymbol{a}|)$. After a lengthy calculation, this yields

$$\int_0^1 \widehat{\rho}(\boldsymbol{a})^{1-s}\,\tau(b_0,\boldsymbol{b})\,\widehat{\rho}(\boldsymbol{a})^s\,ds = \int_0^1 \tau\big(\tfrac{1}{2}b_0 + \boldsymbol{a}\cdot\boldsymbol{b},\ g(\boldsymbol{a},\boldsymbol{b},s)\big)\,ds \quad \text{(A.59)}$$

with

$$g(\boldsymbol{a},\boldsymbol{b},s) = \frac{b_0}{2}\boldsymbol{a} + \frac{a_+^s a_-^{1-s} + a_+^{1-s}a_-^s}{2}\boldsymbol{b} + \frac{2\cdot\frac{1}{2} - (a_+^s a_-^{1-s} + a_+^{1-s}a_-^s)}{2|\boldsymbol{a}|^2}(\boldsymbol{a}\otimes\boldsymbol{a})\boldsymbol{b}. \quad \text{(A.60)}$$

Due to the linearity of τ we can just integrate the function g in (A.59) w.r.t. s. Due to

$$\begin{aligned}\int_0^1 a_+^s a_-^{1-s} + a_+^{1-s}a_-^s\,ds &= \frac{a_+ - a_-}{\log a_+ - \log a_-} + \frac{a_- - a_+}{\log a_- - \log a_+}\\ &= 2\,\frac{2|\boldsymbol{a}|}{\log(1/2+|\boldsymbol{a}|) - \log(1/2 - |\boldsymbol{a}|)} = 2\lambda(|\boldsymbol{a}|) \quad \text{(A.61)}\end{aligned}$$

this yields (A.57).

With these examples, the "miracle relations" $\big[\mathcal{C}_\rho A,\ \log\rho\big] = \big[A,\ \rho\big] = \mathcal{C}_\rho\big[A,\ \log\rho\big]$ from (2.62) can easily be checked. We have

$$\begin{aligned}\mathcal{C}_{\widehat{\rho}(\boldsymbol{a})}\big[\tau(q_0,\mathbf{q}),\log\widehat{\rho}(\boldsymbol{a})\big] &= \mathcal{C}_{\widehat{\rho}(\boldsymbol{a})}\tau\Big(0,\ -2i\mathbf{q}\times\frac{\boldsymbol{a}}{\lambda(|\boldsymbol{a}|)}\Big)\\ &= \tau\Big(0,\ -2i\mathbf{q}\times\boldsymbol{a}\Big) = \big[\tau(q_0,\mathbf{q}),\ \widehat{\rho}(\boldsymbol{a})\big]. \quad \text{(A.62)}\end{aligned}$$

Here we have used the relation $(\boldsymbol{a}\otimes\boldsymbol{b})\boldsymbol{c} = (\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}$ which implies

$$(\boldsymbol{a}\otimes\boldsymbol{a})(\mathbf{q}\times\boldsymbol{a}) = \big(\boldsymbol{a}\cdot(\mathbf{q}\times\boldsymbol{a})\big)\boldsymbol{a} = 0. \quad \text{(A.63)}$$

Moreover, in Bloch coordinates it is straight forward to show that $\mathsf{C}_{\boldsymbol{a}}$ is a positive semi-definite operator on $\mathbb{R}^3_{\boldsymbol{a}}$.

[7] We note that also μ can be continuously extended to the set $\{\boldsymbol{a} \in \mathcal{A} :\ |\boldsymbol{a}| = 0\}$.

Lemma A.2.3. *For all $\boldsymbol{a} \in \mathbb{R}^3_{\boldsymbol{a}}$ the matrix $\mathsf{C}_{\boldsymbol{a}} \in \mathbb{R}^{3\times 3}_{\boldsymbol{a}}$ is positive semi-definite, i.e. for the eigenvalues $\nu_1(\boldsymbol{a})$, $\nu_2(\boldsymbol{a})$ and $\nu_3(\boldsymbol{a})$ we have*

$$\forall \boldsymbol{a} \in \mathbb{R}^3_{\boldsymbol{a}} : \qquad \nu_j(\boldsymbol{a}) \geq 0, \qquad j \in \{1,2,3\}. \tag{A.64}$$

Proof. In the following we write λ instead of $\lambda(|\boldsymbol{a}|)$ and μ instead of $\mu(|\boldsymbol{a}|)$. The characteristic polynomial $p(\nu)$ of $\mathsf{C}_{\boldsymbol{a}}$ is given by

$$\begin{aligned}
p(\nu) &= \big(\lambda + a_1^2\,\mu - \nu\big)\big(\lambda + a_2^2\,\mu - \nu\big)\big(\lambda + a_3^2\ - \nu\big) + 2\mu^3\big(a_1^2\,a_2^2\,a_3^2\big) \\
&\quad - \Big(\mu^2\,a_1^2\,a_3^2\big(\lambda + \mu\,a_2^2 - \nu\big) + \mu^2\,a_3^2\,a_2^2\big(\lambda + \mu\,a_1^2 - \nu\big) + \mu^2\,a_2^2\,a_1^2\big(\lambda + \mu\,a_3^2 - \nu\big)\Big) \\
&= \big(\lambda + a_1^2\,\mu - \nu\big)\big(\lambda + a_2^2\,\mu - \nu\big)\big(\lambda + a_3^2\ - \nu\big) + 2\mu^3\big(a_1^2\,a_2^2\,a_3^2\big) \\
&\quad - 3\mu^3\big(a_1^2\,a_2^2\,a_3^2\big) - \mu^2\Big(a_1^2\,a_3^2\big(\lambda - \nu\big) + a_2^2\,a_3^2\big(\lambda - \nu\big) + a_1^2\,a_2^2\big(\lambda - \nu\big)\Big) \\
&= \big(\lambda + a_1^2\,\mu - \nu\big)\big(\lambda + a_2^2\,\mu - \nu\big)\big(\lambda + a_3^2\ - \nu\big) - \mu^3\big(a_1^2\,a_2^2\,a_3^2\big) \\
&\quad - \mu^2\big(\lambda - \nu\big)\big(a_1^2\,a_3^2 + a_2^2\,a_3^2 + a_1^2\,a_2^2\big).
\end{aligned}$$

A rather lengthy calculation yields

$$p(\nu) = -\nu^3 + \big(3\lambda + \mu|\boldsymbol{a}|^2\big)\nu^2 - \big(3\lambda^2 + 2\lambda\mu|\boldsymbol{a}|^2\big)\nu + \lambda^2|\boldsymbol{a}|^2. \tag{A.65}$$

Finally, inserting $\mu = \frac{1-2\lambda}{2|\boldsymbol{a}|^2}$ yields

$$p(\nu) = -\nu^3 + \big(2\lambda + \tfrac{1}{2}\big)\nu^2 - \lambda\big(\lambda + 1\big)\nu + \lambda^2\big(\tfrac{1}{2} - \lambda\big). \tag{A.66}$$

Due to $0 \leq \lambda \leq 1/2$, we have for the coefficients $a(\boldsymbol{a}) := 2\lambda + \frac{1}{2}$, $b(\boldsymbol{a}) := \lambda(\lambda+1)$ and $c(\boldsymbol{a}) := \lambda^2(\frac{1}{2} - \lambda)$ that the following holds.

$$\forall \boldsymbol{a} \in \mathbb{R}^3_{\boldsymbol{a}} : \qquad a(\boldsymbol{a}) = 2\lambda + \tfrac{1}{2} \geq 0, \qquad b(\boldsymbol{a}) = \lambda(\lambda+1) \geq 0, \qquad c(\boldsymbol{a}) = \lambda^2(\tfrac{1}{2} - \lambda) \geq 0.$$

Therefore, the zeros of p are non-negative. To see this, consider an arbitrary polynomial $f(x) = x^3 - ax^2 + bx - c$ with a, b, $c \geq 0$ and assume that $x_0 < 0$ is a zero of f. This leads to a contradiction, because we have $0 = x_0^3 - ax_0^2 + bx_0 - c < 0$.

This implies that for all $\boldsymbol{a} \in \mathbb{R}^3_{\boldsymbol{a}}$ the eigenvalues $\nu_1(\boldsymbol{a})$, $\nu_2(\boldsymbol{a})$ and $\nu_3(\boldsymbol{a})$ of $\mathsf{C}_{\boldsymbol{a}}$ are non-negative. Thus, for all $\boldsymbol{a} \in \mathbb{R}^3_{\boldsymbol{a}}$ the matrix $\mathsf{C}_{\boldsymbol{a}}$ is positive semi-definite. □

We end this discussion by noting that in [Ött10] equivalent considerations have been made for a similar transformation $\mathcal{O}(a_0, \boldsymbol{a}) = \frac{1}{2}\tau(a_0, a_2, a_3, a_1)$.

A.3. Fourier Transformation and Pseudo-differential Operators

In this section we give a brief introduction to the Fourier transformation and a summary on relevant results from the theory of pseudo-differential operators (PDOs). We restrict our presentation to the case of $\mathbb{C}$-valued functions stressing that if not declared otherwise, all functions in this section are $\mathbb{C}$-valued. In the case of $\mathbb{C}^N$-valued functions, the following calculations have to be done component wise. For more comprehensive studies we refer to [AlG07], [EgS97], [Gra08] and [Ste93].

We begin with introducing the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ consisting of smooth rapidly decaying functions.

Definition A.3.1 (Schwartz space, [Gra08, Remark 2.2.4])**.** *For $d \in \mathbb{N}$, the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ consists of the subset of $C^\infty(\mathbb{R}^d)$ such that for all $u \in \mathcal{S}(\mathbb{R}^d)$ the following property is satisfied*

$$\forall\, \alpha \in \mathbb{N}_0^d,\ \forall\, N \in \mathbb{N},\ \exists\, C_{\alpha,N} \quad \text{such that} \quad \sup_{x\in\mathbb{R}^d} \big|\partial^\alpha u(x)\big| \le C_{\alpha,N}\big(1+|x|\big)^{-N}. \tag{A.67}$$

Obviously, we have $C_c^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$ (see [Gra08, p. 109]) and $\mathcal{S}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ for all $p \ge 1$ (see [Wer11, p. 214]). We denote the dual of the Schwartz space with $\mathcal{S}'(\mathbb{R}^d)$ and call this space the *space of tempered distributions*. Next, we give the definition of the Fourier transform and list some of its properties. For more details see [Gra08], [AlG07], [Ste93] or [EgS97].

Definition A.3.2. *For a Schwartz function $u \in \mathcal{S}(\mathbb{R}^d)$, we define the Fourier transformation $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}(\mathbb{R}^d)$ by $\mathcal{F} : u \longmapsto \widehat{u}$. The function $\widehat{u}$ satisfies the formula*

$$\widehat{u}(\xi) := \int_{\mathbb{R}^d_x} u(x)\, e^{-2i\pi x\cdot\xi}\, dx \tag{A.68}$$

and we call $\widehat{u} := \mathcal{F}[u]$ the Fourier transform of u.

The Fourier transformation $\mathcal{F}$ is a homeomorphism on the Schwartz space[8] $\mathcal{S}(\mathbb{R}^d)$ with inverse $\mathcal{F}^{-1} : u \longmapsto u^\vee$ where $u^\vee$ satisfies the formula

$$u^\vee(x) := \int_{\mathbb{R}^d_\xi} u(\xi)\, e^{2i\pi\xi\cdot x}\, d\xi. \tag{A.69}$$

We call the operator $\mathcal{F}^{-1}$ the *inverse Fourier transformation* and we call $u^\vee := \mathcal{F}^{-1}[u]$ the *inverse Fourier transform* of u. Obviously the Fourier transformation $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ are linear. Moreover, for $u \in \mathcal{S}(\mathbb{R}^d)$ we have $\mathcal{F}^{-1}[\mathcal{F}u] = u = \mathcal{F}[\mathcal{F}^{-1}u]$.

Since the space $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, we can extend the Fourier transform to an operator on $L^2(\mathbb{R}^d)$ and the same holds for the inverse Fourier transformation. We also denote these operators by $\mathcal{F}$ and $\mathcal{F}^{-1}$, respectively, and call them *Fourier-Plancherel trans-*

[8] The Fourier transformation does not map the space $\mathcal{D}(\mathbb{R}^d) = C_c^\infty(\mathbb{R}^d)$ into itself.

formation (or its inverse, respectively)[9]. On the space $L^2(\mathbb{R}^d)$ the Fourier-Plancherel transformation and its inverse are bijections. Furthermore, the Fourier inversion formula

$$u = \mathcal{F}^{-1} \circ \mathcal{F}u = \mathcal{F} \circ \mathcal{F}^{-1}u \tag{A.70}$$

also holds for the Fourier-Plancherel transformation. In general, the Fourier-Plancherel transformation on $L^2(\mathbb{R}^d)$ cannot be expressed by (A.68) since the resulting integral does not necessarily converge absolutely. However, for functions $u \in (L^1 \cap L^2)(\mathbb{R}^d)$, the integrals (A.68), (A.69) actually do converge[10] In this case we have $u(x) = (\widehat{u})^\vee(x)$ for a.e. $x \in \mathbb{R}^d$. The Fourier transformation and its inverse are often introduced in similar ways to (A.68), (A.69) (see [AlG07], [EgS97], [Dob06]). Depending on the context one or the other formulation may be advantageous. We chose the unitary version with a symmetry in the convolution.

Without proof, we list the following properties of the Fourier transformation in the next proposition.[11] We stress that the statements of this proposition crucially depend on the choice of the Fourier transformation.

Proposition A.3.3. *Let u, $v \in L^2(\mathbb{R}^d)$ be given and let x, $\xi \in \mathbb{R}^d$ as well as $b \in \mathbb{C}$. Then, the following holds*

(i) $\mathcal{F}[u * v] = \mathcal{F}u \cdot \mathcal{F}v, \qquad \mathcal{F}[u \cdot v] = \mathcal{F}u * \mathcal{F}v$

(ii) $\mathcal{F}^{-1}[\mathcal{F}[u]] = u \qquad \Longrightarrow \qquad \mathcal{F}^{-1}[u * v] = u^\vee \cdot v^\vee, \qquad \mathcal{F}^{-1}[u \cdot v] = u^\vee * v^\vee$

(iii) $\big\|u\big\|_{L^2(\mathbb{R}^d)} = \big\|\mathcal{F}[u]\big\|_{L^2(\mathbb{R}^d)} = \big\|\mathcal{F}^{-1}[u]\big\|_{L^2(\mathbb{R}^d)}$

(iv) $\int_{\mathbb{R}^d} u(x) \cdot \widehat{v}(x)\, dx = \int_{\mathbb{R}^d} \widehat{u}(x) \cdot v(x)\, dx$

(v) $\int_{\mathbb{R}^d} u(x) \cdot \overline{v(x)}\, dx = \int_{\mathbb{R}^d} \widehat{u}(\xi) \cdot \overline{\widehat{v}(\xi)}\, d\xi$

(vi) $\int_{\mathbb{R}^d} u(x) \cdot v(x)\, dx = \int_{\mathbb{R}^d} \widehat{u}(x) \cdot v^\vee(x)\, dx.$

For u, $v \in \mathcal{S}(\mathbb{R}^d)$ and an arbitrary multi-index $\alpha \in \mathbb{N}_0^d$ it holds

(i) $\|\widehat{u}\|_{L^\infty(\mathbb{R}^d)} \leq \|u\|_{L^1(\mathbb{R}^d)}$

(ii) $\mathcal{F}(\partial_x^\alpha u)(\xi) = (2\pi i \xi)^\alpha \mathcal{F}(u)(\xi)$

(iii) $\partial_\xi^\alpha \mathcal{F}(u)(\xi) = \mathcal{F}\big((-2\pi i x)^\alpha u(x)\big)(\xi).$

By duality we can also define the Fourier transformation of a tempered distribution $T \in \mathcal{S}'(\mathbb{R}^d)$.

Definition A.3.4. *For $T \in \mathcal{S}'(\mathbb{R}^d)$ we define the Fourier transform $\widehat{T}$ and the inverse Fourier transform $T^\vee$ of T by setting*

$$\widehat{T}[u] := T[\widehat{u}], \qquad T^\vee[u] := T[u^\vee], \qquad \text{for all } u \in \mathcal{S}(\mathbb{R}^d). \tag{A.71}$$

[9] In the following we will also call the Fourier-Plancherel transformation *Fourier transformation.*

[10] See [Gra08, Sec. 2.2.4, p. 103].

[11] For more details and a proof see [Gra08, Prop. 2.2.11, Th. 2.2.14 and Prop. 2.3.22].

The Fourier transformation of a tempered distribution enjoys the properties from Proposition A.3.3 and opens the possibility to define Sobolev spaces with real-valued regularity

Definition A.3.5. *For $s \in \mathbb{R}$ we define the space $H^s(\mathbb{R}^d)$ as the subspace of $\mathcal{S}'(\mathbb{R}^d)$ such that for all $u \in H^s(\mathbb{R}^d)$ we have $\widehat{u} \in L^2_{\text{loc}}(\mathbb{R}^d)$ and*

$$\|u\|^2_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d_\xi} (1 + |\xi|^2)^s |\widehat{u}(\xi)|^2 \, d\xi < \infty. \tag{A.72}$$

For $s \in \mathbb{N}$, the space $H^s(\mathbb{R}^d)$ coincides with the space $W^{s,2}(\mathbb{R}^d)$ from Definition A.1.15 and the corresponding norms are equivalent.[12]

Pseudo-differential Operators

Formally, for a given function $a : \mathbb{R}^d_x \times \mathbb{R}^d_\xi \longrightarrow \mathbb{C}$, we can define the operator A acting on a function $u : \mathbb{R}^d \longrightarrow \mathbb{C}$ by

$$\mathsf{A}u := \mathcal{F}^{-1}[a \cdot \widehat{u}], \qquad \mathsf{A}[u](x) := (2\pi)^{-d} \int_{\mathbb{R}^d_\xi} a(x,\xi)\widehat{u}(\xi)\, e^{i\pi\xi\cdot x}\, d\xi. \tag{A.73}$$

Clearly, this definition also makes sense for every function $u : \mathbb{R}^d \longrightarrow \mathbb{C}^N$. Proceeding formally, we note that the function a is called the *symbol* of the *pseudo-differential operator* A. It is common to denote the operator A defined above by $a(x, D)$. As in [Ste93], if the symbol a is independent of x, we call the operator A a (Fourier) *multiplier operator* and we denote A with $a(D)$. In contrast, if the symbol a is independent of ξ, the operator A is called a *multiplication operator*.

The immediate question is of course, under which conditions such an operator is well defined. In this brief summary we content ourselves with partly answering this question for operators with symbols taken from the Hörmander classes S^m acting on the spaces $\mathcal{S}(\mathbb{R}^d)$ and $H^s(\mathbb{R}^d)$.

Definition A.3.6 (Hörmander classes, [AlG07, p. 20 and p. 25])**.** *Let $m \in \mathbb{R}$ and let α, $\beta \in \mathbb{N}^d_0$ be multi-indices. The Hörmander Class S^m is defined as the following set*

$$S^m := \Big\{ a \in C^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_\xi) : \forall\alpha, \forall\beta \text{ it holds } \big|\partial^\beta_x \partial^\alpha_\xi a(x,\xi)\big| \le C_{\alpha,\beta}(1 + |\xi|)^{m-|\alpha|} \Big\}. \tag{A.74}$$

We set $S^{-\infty} := \cap_m S^m$. An element $a \in S^m$ is called a symbol of order m. For $a \in S^m$ the operator $\mathsf{A} = a(x, D)$ defined by (A.73) *is called the pseudo-differential operator with symbol a. A pseudo-differential operator is said to be of order m if its symbol belongs to S^m.*

Such classes were introduced by Hörmander in [Hör67]. More general definitions of Hörmander class symbols can for example be found in [Tay81, Ch. II].

[12] See for example [Dob06, Sec. 9.4.].

Note, in particular, that if a symbol a belongs to the class S^m, then, it also belongs to every other class S^n with $n \geq m$. In [BeS07, Ch. C.1] Hörmander classes for $\mathbb{C}^N$-valued functions u are introduced the following way.

$$\mathbf{S}^m := \Big\{ a \in C^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_\xi; \mathbb{C}^{N\times N}) : \forall \alpha, \forall \beta \text{ it holds } \big|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\big| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|} \Big\}.$$

We have the following continuity result.

Lemma A.3.7 ([Ste93, Prop. 5, p. 251]). *For $m \in \mathbb{N}$ let a symbol $a \in S^m$ be given and let $\mathsf{A} = a(x, D)$ denote the corresponding PDO. Then, the operator $\mathsf{A} : H^{s+m}(\mathbb{R}^d) \longrightarrow H^s(\mathbb{R}^d)$ is continuous.*

For pure *multiplier* or pure *multiplication* operators A, B, it is clear that we have[13] $\mathsf{AB} = \mathsf{BA}$. We stress that for general pseudo-differential operators this is not the case, but the following holds.

Proposition A.3.8. *For $m_a, m_b \in \mathbb{R}$ let $a \in S^{m_a}$ and $b \in S^{m_b}$ be given. With A, B we denote the corresponding pseudo-differential operators. Then the commutator $[A, B] = AB - BA$ of the two operators is a pseudo-differential operator of order $m = m_a + m_b - 1$. Furthermore, its symbol c satisfies*

$$c(x,\xi) = \frac{1}{2\pi i} \sum_{j=1}^{d} \Big(\frac{\partial a}{\partial \xi_j} \frac{\partial b}{\partial x_j} - \frac{\partial a}{\partial x_j} \frac{\partial b}{\partial \xi_j} \Big) \qquad \textit{mod } S^{m_a+m_b-2}. \tag{A.75}$$

We stated Proposition A.3.8 for completeness but we will not use it. We note that equality (A.75) depends on the choice of the Fourier transformation. For different choices of the Fourier transformation, the result can be found in [AlG07, Cor. 4.1, p. 29] and in [EgS97, Cor. 18, p. 37]. See also [Ste93, Th. 2, p. 237] where our choice of the Fourier transformation is used. Moreover, [BeS07, Theorem C.3 (ii)] states that the above result is also valid, if one of the symbols is from the symbol class $\mathbf{S}^m$ and the other is from the symbol class S^m.

Important examples are multiplier operators whose symbols have compact support. Namely, we consider symbols satisfying the next hypothesis.

Hypothesis A.3.9. *Let $\chi \in C_c^\infty(\mathbb{R}^d)$ be a function satisfying the following properties.*

(i) For all $x \in \mathbb{R}^d$ the function χ satisfies $0 \leq \chi(x) \leq 1$,

(ii) The function χ satisfies $\chi(x) = \begin{cases} 1, & \text{if } |x| \leq 1/2 \\ 0, & \text{if } |x| \geq 1. \end{cases}$

(iii) For all $x \in \mathbb{R}^d$ the function χ satisfies $\chi(-x) = \chi(x)$.

[13] In the case of $\mathbb{C}^{N\times N}$-valued symbols the corresponding pure multiplication or pure multiplier operators do of course not necessarily commute.

Due to the next lemma it makes sense to call Fourier multiplier operators with symbols that satisfy Hypothesis A.3.9 *smoothing operators*

Lemma A.3.10 (Bernstein type lemma, [BCD11, p. 52])**.** *Let $r > 0$, $d \in \mathbb{N}$ and define $B_r(0) := \{\xi \in \mathbb{R}^d : |\xi| \leq r\}$. There exists a constant $C_{\mathrm{bsl}} > 0$ such that for every $k \in \mathbb{N}$, every couple $(p, q) \in [1, \infty]^2$ with $p \leq q$ and every function $u \in L^p(\mathbb{R}^d)$ with* $\operatorname{supp} \widehat{u} \subset \lambda B_r(0)$*, we have*

$$\sup_{|\alpha|=k} \|\partial_x^\alpha u\|_{L^q(\mathbb{R}^d)} \leq C_{\mathrm{bsl}}^{k+1} \lambda^{k+d(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p(\mathbb{R}^d)}. \tag{A.76}$$

Important Examples of PDOs

1. For $s \in \mathbb{R}$ we introduce the symbol

$$\lambda^s(\xi) := (1 + |\xi|^2)^{s/2}. \tag{A.77}$$

With this, we can define the spaces $H^s(\mathbb{R}^d)$ in the following way

$$H^s(\mathbb{R}^d) := \big\{u \in \mathcal{S}'(\mathbb{R}^d) : \lambda^s(D)\, u \in L^2(\mathbb{R}^d)\big\}. \tag{A.78}$$

Moreover, it is straight forward to see that the corresponding multiplier operator $\Lambda^s := \lambda^s(D)$ is an isometry from the space $H^{s+t}(\mathbb{R}^d)$ to $H^t(\mathbb{R}^d)$ for all $t \in \mathbb{R}$. In particular, for $s \geq 0$ we have

$$\Lambda^s : H^s(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d) \qquad \text{with } \big\|\Lambda^s\big\|_{\mathcal{L}(H^s, L^2)} = 1 \tag{A.79a}$$

$$\Lambda^{-s} : L^2(\mathbb{R}^d) \longrightarrow H^s(\mathbb{R}^d) \qquad \text{with } \big\|\Lambda^{-s}\big\|_{\mathcal{L}(L^2, H^s)} = 1. \tag{A.79b}$$

2. For a function χ satisfying Hypothesis A.3.9 and for arbitrary $\nu \geq 1$, we set $\chi_\nu(\xi) := \chi(\xi/\nu)$. Then, we define the Fourier multiplier operator[14]

$$\mathsf{S}^\nu := \chi_\nu(D), \quad \text{i.e.} \quad \mathsf{S}^\nu u = \mathcal{F}^{-1}[\chi_\nu \cdot \widehat{u}]. \tag{A.80}$$

Obviously, for every $u \in L^2(\mathbb{R}^d)$, the image $\mathsf{S}^\nu u$ is an element of the space

$$L^2_\nu(\mathbb{R}^d) := \big\{u \in L^2(\mathbb{R}^d) |\ \operatorname{supp}(\mathcal{F}(u)) \subset \overline{B_\nu(0)}\big\}. \tag{A.81}$$

In particular, Lemma A.3.10 yields that for every fixed $\nu \geq 1$, we have the inclusion

$$L^2_\nu(\mathbb{R}^d) \subset \bigcap_{s \in \mathbb{R}} H^s(\mathbb{R}^d). \tag{A.82}$$

3. (See also [AlG07, Example 2, p. 20].) Let $a \in C^\infty(\mathbb{R}^d \setminus \{0\})$ be a positively homogeneous function of degree m and let χ be a function that satisfies Hypothesis A.3.9. Then, the function $\widetilde{a}(\xi) := \big(1 - \chi(\xi)\big) a(\xi)$ is a symbol of order m.

[14]The $\cdot$ is to be understood as the multiplication of $\widehat{u}$ with the scalar-valued function χ_ν. In particular, for $\mathbb{C}^N$-valued functions u this multiplication has to be done component wise.

Remark A.3.11. *In Section A.4 the positively homogeneous $C^\infty(\mathbb{R}^3;\mathbb{C}^{3\times 3})$-functions of degree zero $\pi_\parallel$ and $\pi_\perp$ are introduced. They are the symbols of the projectors $\mathsf{P}_\parallel$ and $\mathsf{P}_\perp$. For $\xi = 0$ these symbols are singular. Therefore, they do not belong to any of the Hörmander symbol classes. Nevertheless, for every function χ satisfying Hypothesis A.3.9, the functions $\big(1-\chi(\xi)\big)\pi_\parallel(\xi)$ and $\big(1-\chi(\xi)\big)\pi_\perp(\xi)$ are symbols of the class $\mathbf{S}^0$.*

For $m \in \mathbb{R}$, we call a set $\mathrm{A} \subset S^m$ a *bounded set of symbols of order m* if for all multi-indices α, $\beta \in \mathbb{N}_0^d$ there exist constants $C_{\alpha,\beta}$ independent of the choice of $a \in \mathrm{A}$ such that

$$\sup_{a\in\mathrm{A}} \big|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\big| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}. \tag{A.83}$$

The family of pseudo-differential operators $\{a(D) : a \in \mathrm{A}\}$ of order m is said to be bounded, if the corresponding set of symbols $\mathrm{A} \subset S^m$ is bounded. In particular, for symbols of order zero we have the following lemma.

Lemma A.3.12. *The product of two symbols a_1, $a_2 \in S^0$ is also a symbol of order zero. In particular, if $\mathrm{A} \subset S^0$ is bounded, then, for a given $b \in S^0$ the sets $b\mathrm{A} := \{b\,a : a \in \mathrm{A}\}$ and $\mathrm{A}b := \{a\,b : a \in \mathrm{A}\}$ are also bounded sets of symbols of order zero.*

We skip the simple proof of this lemma and refer to [AlG07, Lemma 2.1.1, p. 21].
Next, we list some properties of the smoothing operator S^ν from (A.80).

Lemma A.3.13. *For every $\nu \ge 1$, the operator S^ν has the following properties.*

(i) *For all $p \in [1,\infty)$ the operator $\mathsf{S}^\nu : L^p(\mathbb{R}^d) \longrightarrow L^p(\mathbb{R}^d)$ is continuous with norm $\|\mathsf{S}^\nu\|_{\mathcal{L}(L^p,L^p)} \le \|\mathcal{F}(\chi)\|_{L^1}$.*

(ii) *$\mathsf{S}^\nu : L^2(\mathbb{R}^d) \longrightarrow L^\infty(\mathbb{R}^d)$ is continuous with norm $\|\mathsf{S}^\nu\|_{\mathcal{L}(L^2,L^\infty)} \le (2\nu)^{d/2}$.*

(iii) *$\mathsf{S}^\nu : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ is continuous with norm $\|\mathsf{S}^\nu\|_{\mathcal{L}(L^2,L^2)} \le 1$.*

(iv) *For all $u \in L^2(\mathbb{R}^d)$ it holds $\|\mathsf{S}^\nu u - \mathsf{Id} u\|_{L^2} \longrightarrow 0$ as $\nu \to \infty$ and for all $\nu \ge 1$ it holds $\mathsf{S}^\nu = (\mathsf{S}^\nu)^*$ in the space L^2.*

(v) *S^ν is a pseudo-differential operator of degree $-\infty$; in view of Lemma A.3.10 we have $\mathsf{S}^\nu : L^2(\mathbb{R}^d) \longrightarrow H^\infty(\mathbb{R}^d)$.*

(vi) *The operators S^ν commute with the operators[15] $\mathsf{P}_\parallel$ and $\mathsf{P}_\perp$. Moreover, for $m \in \mathbb{N}$ the operators S^ν and ∂_x commute as operators acting on $H^m(\mathbb{R}^3)$.*

(vii) *The symbols $\{\chi_\nu\}_{\nu\ge 1}$ of the family of operators $\{\mathsf{S}^\nu\}_{\nu \ge 1}$ form a bounded set of symbols of degree zero.*

Proof. (i) Due to [Gra08, Prop. 2.5.14, p. 145] it holds $\|\mathsf{S}^\nu\|_{\mathcal{L}(L^p,L^p)} = \|\mathsf{S}^1\|_{\mathcal{L}(L^p,L^p)}$. Since χ is an even function, we have $\mathcal{F}^{-1}(\chi) = \mathcal{F}(\chi)$. The convolution estimate (A.34) yields

$$\|S^1 u\|_{L^p} = \|\mathcal{F}^{-1}(\chi\cdot\mathcal{F}(u))\|_{L^p} = \|\mathcal{F}^{-1}(\chi) * u\|_{L^p} \le \|\mathcal{F}^{-1}(\chi)\|_{L^1}\|u\|_{L^p} = \|\widehat{\chi}\|_{L^1}\|u\|_{L^p}.$$

[15]The operators $\mathsf{P}_\parallel$ and $\mathsf{P}_\perp$ are introduced in Section A.4

(ii) On the one hand, we have

$$\|\mathsf{S}^\nu\|_{\mathcal{L}(L^2,L^\infty)} = \sup_{u\in L^2} \frac{\|\mathsf{S}^\nu u\|_{L^\infty}}{\|u\|_{L^2}}. \tag{A.84}$$

Due to the definition of the L^∞-norm and the operator S^ν, we get on the other hand

$$\begin{aligned}\|\mathsf{S}^\nu u\|_{L^\infty} &= \sup_{x\in\mathbb{R}^d} \Big\{ \int_{\mathbb{R}^d_\xi} e^{2i\pi x\cdot\xi} \chi_\nu(\xi)\widehat{u}(\xi)\, d\xi \Big\} \\ &\le \|\chi_\nu\|_{L^2(\mathbb{R}^d)} \|\widehat{u}\|_{L^2(\mathbb{R}^d)} \le (2\nu)^{d/2} \|u\|_{L^2(\mathbb{R}^d)}.\end{aligned} \tag{A.85}$$

This yields the assertion.

(iii) For $u \in L^2_{\nu/2}(\mathbb{R}^d) \subset L^2_\nu(\mathbb{R}^d)$ it holds $\chi_\nu \widehat{u} = \widehat{u}$. Thus, for $u \in L^2_{\nu/2}(\mathbb{R}^d)$ we have

$$\|\mathsf{S}^\nu u\|_{L^2} = \|\mathcal{F}^{-1}(\chi_\nu \mathcal{F} u)\|_{L^2} = \|\mathcal{F}^{-1}(\mathcal{F}u)\|_{L^2} = \|u\|_{L^2} \tag{A.86}$$

and we can infer $\|\mathsf{S}^\nu\|_{\mathcal{L}(L^2,L^2)} \ge 1$. On the other hand, for arbitrary $u \in L^2$, we get the following estimate from Young's inequality

$$\|\mathsf{S}^\nu u\|_{L^2} = \|\mathcal{F}^{-1}(\chi_\nu \mathcal{F}u)\|_{L^2} = \|\chi_\nu \mathcal{F}u\|_{L^2} \le \|\chi_\nu\|_{L^\infty} \|\mathcal{F}u\|_{L^2} = 1\, \|u\|_{L^2}, \tag{A.87}$$

so that we can infer $\|\mathsf{S}^\nu\|_{\mathcal{L}(L^2,L^2)} \le 1$. Thus, we have shown that $\|\mathsf{S}^\nu\|_{\mathcal{L}(L^2,L^2)} = 1$ holds. For arbitrary u, $v \in L^2$ and $\nu \ge 1$ we have by Parseval's identity the relation

$$\begin{aligned}\big(u, \mathsf{S}^\nu v\big)_{L^2} &= \int_{\mathbb{R}^d} u(x) \cdot \overline{\mathcal{F}^{-1}[\chi_\nu \cdot \widehat{v}](x)}\, dx = \int_{\mathbb{R}^d} \widehat{u}(\xi) \cdot \overline{\chi_\nu \cdot \widehat{v}(\xi)}\, d\xi \\ &= \int_{\mathbb{R}^d} \chi_\nu \cdot \widehat{u}(\xi) \cdot \overline{\widehat{v}(\xi)}\, d\xi = \int_{\mathbb{R}^d} \mathcal{F}^{-1}[\chi_\nu \cdot \widehat{u}](x) \cdot \overline{v(x)}\, dx = \big(\mathsf{S}^\nu u, v\big)_{L^2}.\end{aligned} \tag{A.88}$$

Thus, we have shown that for all $\nu \ge 1$ it holds $(\mathsf{S}^\nu)^* = \mathsf{S}^\nu$.

(iv) Since for arbitrary $u \in L^2(\mathbb{R}^3)$ and arbitrary $\epsilon > 0$, there exists $R > 0$, such that $\|u - \chi_{B_R(0)} u\|_{L^2(\mathbb{R}^3)} \le \epsilon$, the result is immediate. (v) Is obvious. (vi) All considered pseudo-differential operators are pure multiplier operators and the operators S^ν are scalar valued this yields the claim.

(vii) Since $\chi \in C^\infty_c(\mathbb{R}^d)$, we obviously have that for every muli-indicex $\alpha \in \mathbb{N}^d_0$ there exists a constant C_α such that $|\partial^\alpha_\xi \chi(\xi)| \le C_\alpha (1 + |\xi|)^{-|\alpha|}$. With $u := \xi/\nu$ we have

$$\begin{aligned}\sup_{\nu\ge1} \big|\partial^\alpha_\xi \chi_\nu(\xi)\big| &= \sup_{\nu\ge1} \big|\partial^\alpha_\xi \chi(\xi/\nu)\big| = \sup_{\nu\ge1} \Big(\big|\partial^\alpha_u \chi(u)\big|\, \nu^{-|\alpha|} \Big) \\ &\le \sup_{\nu\ge1} \Big(C_\alpha \big(1 + |\xi/\nu|\big)^{-|\alpha|} \nu^{-|\alpha|} \Big) = \sup_{\nu\ge1} \Big(C_\alpha \big(\nu + |\xi|\big)^{-|\alpha|} \Big) \\ &\le C_\alpha \big(1 + |\xi|\big)^{-|\alpha|}.\end{aligned} \tag{A.89}$$

This proves the claim. □

A Rellich-type Lemma

We end this section by proving a generalized version of the Rellich-type lemma stated in [JMR00a, Lemma 4.3] and [Dum05, Lemma 2.3].

Theorem A.3.14 (Rellich-type lemma)**.** *Let a bounded family of pure multiplier symbols $\{m_\nu\}_{\nu\geq 1} \subset S^0$, i.e. $m_\nu = m_\nu(\xi)$, and a function $\phi \in C_c^\infty(\mathbb{R}^d)$ be given. We introduce the operator M_ϕ by*

$$\mathsf{M}_\phi[u] := \phi \cdot u \tag{A.90}$$

and define the family of operators $\{T_\phi^\nu\}_{\nu\geq 1}$ by

$$T_\phi^\nu := \big[\mathsf{M}_\phi, m_\nu(D)\big]. \tag{A.91}$$

Then, the following holds. For all $\{u_\nu\}_{\nu\geq 1} \subset L^2(\mathbb{R}^d)$ with

$$u_\nu \longrightarrow 0 \qquad \textit{weakly in } L^2(\mathbb{R}^d) \qquad \textit{as } \nu \to \infty \tag{A.92}$$

we have the strong convergence

$$T_\phi^\nu u_\nu \longrightarrow 0 \qquad \textit{strongly in } L^2(\mathbb{R}^d) \qquad \textit{as } \nu \to \infty. \tag{A.93}$$

Note that for each $\nu_0 \geq 1$ fixed, the statement of the above theorem for the operator $T_\phi^{\nu_0}$ is trivial. The proof of this theorem is fundamentally based on the following lemma.

Lemma A.3.15. *Let the family of operators $\{T_\phi^\nu\}_{\nu\geq 1}$ be defined as in Theorem A.3.14. Then, for all $s \in \mathbb{R}$ it holds*

$$\sup_{\nu\geq 1} \big\|T_\phi^\nu\big\|_{\mathcal{L}(H^s,H^{s+1})} = \sup_{\nu\geq 1} \sup_{\substack{u\in H^s \\ \|u\|_{H^s}\neq 0}} \frac{\big\|T_\phi^\nu\, u\big\|_{H^{s+1}}}{\|u\|_{H^s}} < \infty. \tag{A.94}$$

This implies that the family $\{T_\phi^\nu\}_{\nu\geq 1}$ is a bounded family of pseudo-differential operators of degree -1.

Proof of Theorem A.3.14. Let functions ϕ_1, $\phi_2 \in C_c^\infty(\mathbb{R}^d)$ be given with $\phi\,\phi_1 = \phi$ and $\phi_1\,\phi_2 = \phi_1$ and let

$$\Omega := \operatorname{supp}\phi, \qquad \Omega_1 := \operatorname{supp}\phi_1, \qquad \Omega_2 := \operatorname{supp}\phi_2. \tag{A.95}$$

Obviously, we have $\Omega \subset \Omega_1 \subset \Omega_2$ and all these sets are bounded. Moreover, let an arbitrary sequence $\{u_\nu\}_{\nu\geq 1} \subset L^2(\mathbb{R}^d)$ be given with

$$u_\nu \longrightarrow 0 \qquad \text{weakly in } L^2(\mathbb{R}^d) \qquad \text{as } \nu \to \infty. \tag{A.96}$$

Then, with $\mathsf{M}_{\phi_1}[u] := \phi_1 \cdot u$ we have

$$T_\phi^\nu u_\nu = \big(1-\phi_2\big)\big[\mathsf{M}_{\phi_1}, m_\nu(D)\big]\phi\, u_\nu + \phi_2\, T_\phi^\nu\, u_\nu. \tag{A.97}$$

On the one hand, the sequence $\{\phi_2 T^\nu_\phi u_\nu\}_{\nu\geq 1}$ is bounded in $H^1(\Omega_2)$, since we have the estimate

$$\begin{aligned}\left\|\phi_2 T^\nu_\phi u_\nu\right\|_{H^1(\Omega_2)} &= \left\|\phi_2 T^\nu_\phi u_\nu\right\|_{H^1(\mathbb{R}^d)} \leq \|\phi_2\|_{W^{1,\infty}(\mathbb{R}^d)} \left\|T^\nu_\phi u_\nu\right\|_{H^1(\mathbb{R}^d)} \\ &\leq \|\phi_2\|_{W^{1,\infty}(\mathbb{R}^d)} \left\|T^\nu_\phi\right\|_{\mathcal{L}(L^2,H^1)} \|u_\nu\|_{L^2(\mathbb{R}^d)} \end{aligned} \tag{A.98}$$

and due to the bound $\sup_{\nu\geq 1}\left\|T^\nu_\phi\right\|_{\mathcal{L}(L^2,H^1)} < \infty$ from Lemma A.3.15. This implies that the set $\{\phi_2 T^\nu_\phi u_\nu\}_{\nu\geq 1}$ is compact in both $L^2(\Omega_2)$ and $L^2(\mathbb{R}^d)$ since Ω_2 is bounded. The convergence $u_\nu \longrightarrow 0$ weakly in $L^2(\mathbb{R}^d)$ as $\nu \to \infty$ thus yields the convergence

$$\phi_2\, T^\nu_\phi\, u_\nu \longrightarrow 0 \qquad \text{strongly in } L^2(\mathbb{R}^d) \qquad \text{as } \nu \to \infty. \tag{A.99}$$

On the other hand, the sequence $\{\phi\, u_\nu\}_{\nu\geq 1}$ converges weakly to zero in $L^2(\Omega)$. Therefore, $\phi\, u_\nu \longrightarrow 0$ strongly in $H^{-1}(\Omega)$. Again this needs the boundedness of Ω.[16] Setting $T^\nu_{\phi_1} := \left[\mathsf{M}_{\phi_1}, m_\nu(D)\right]$, we can infer the following from Lemma A.3.15

$$\begin{aligned}\left\|T^\nu_{\phi_1}\phi\, u_\nu\right\|_{L^2(\mathbb{R}^d)} &\leq \left\|T^\nu_{\phi_1}\right\|_{\mathcal{L}(H^{-1},L^2)} \cdot \left\|\phi\, u_\nu\right\|_{H^{-1}(\mathbb{R}^d)} \\ &\leq \left\|T^\nu_{\phi_1}\right\|_{\mathcal{L}(H^{-1},L^2)} \cdot \left\|\phi\, u_\nu\right\|_{H^{-1}(\Omega)} \longrightarrow 0 \qquad \text{as } \nu \to \infty. \end{aligned} \tag{A.100}$$

Thus, we have shown the convergence $T^\nu_\phi\, u_\nu \longrightarrow 0$ strongly in $L^2(\mathbb{R}^d)$ as $\nu \to \infty$. □

The following proof is based on ideas from the proof of [Ste93, Th. 2, p. 237].

Proof of Lemma A.3.15. We only prove the case $s = 0$. The definition of the operator $m_\nu(D)$ yields $\mathcal{F}\big(m_\nu(D)\, u\big)(\xi) = m_\nu(\xi)\cdot u(\xi)$ and due to Proposition A.3.3 we have for all $u \in L^2(\mathbb{R}^d)$

$$\mathcal{F}\big(T^\nu_\phi\, u\big)(\xi) = \big(\widehat{\phi} * (m_\nu \cdot \widehat{u})\big)(\xi) - m_\nu(\xi)\cdot \big(\widehat{\phi} * \widehat{u}\big)(\xi). \tag{A.101}$$

Therefore, with the Sobolev symbol λ^1 from (A.77), for arbitrary $u \in L^2(\mathbb{R}^d)$ we may write $\left\|T^\lambda_\phi\, u\right\|^2_{H^1(\mathbb{R}^d)}$ as

$$\begin{aligned}\left\|T^\lambda_\phi\, u\right\|^2_{H^1(\mathbb{R}^d)} &= \left\|\lambda^1\, T^\lambda_\phi\, u\right\|_{L^2(\mathbb{R}^d)} \\ &= \int_{\mathbb{R}^d_\xi} \left|\lambda^1(\xi)\Big(\big(\widehat{\phi} * (m_\nu\cdot\widehat{u})\big)(\xi) - m_\nu(\xi)\cdot\big(\widehat{\phi} * \widehat{u}\big)(\xi)\Big)\right|^2 d\xi \\ &= \int_{\mathbb{R}^d_\xi} \left|\lambda^1(\xi)\int_{\mathbb{R}^d_\eta} \widehat{u}(\xi-\eta)\,\widehat{\phi}(\eta)\Big(m_\nu(\xi-\eta) - m_\nu(\xi)\Big)\, d\eta\right|^2 d\xi. \end{aligned} \tag{A.102}$$

We denote the jth component of the vector ξ with ξ_j. Due to Taylor's theorem (in the version of [Hil03, Cor. 1, p. 53]), there exists some $\vartheta \in (0,1)$ such that

$$m_\nu(\xi-\eta) - m_\nu(\xi) = -\sum_{j=1}^d \partial_{\xi_j} m_\nu(\xi)\,\eta_j + \frac{1}{2}\sum_{j,k=1}^d \partial_{\xi_j}\partial_{\xi_k} m_\nu(\xi - \vartheta\eta)\,\eta_j\eta_k. \tag{A.103}$$

[16] We cannot argue as for the second term, since the embedding $H^1(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)$ is not compact.

Therefore, for this $\vartheta \in (0,1)$ we may estimate the term in (A.102) by

$$\text{(A.102)} \leq \int_{\mathbb{R}^d_\xi} \Big| \lambda^1(\xi) \int_{\mathbb{R}^d_\eta} \widehat{u}(\xi-\eta)\, \widehat{\phi}(\eta) \Big(\sum_{j=1}^d \partial_{\xi_j} m_\nu(\xi)\, \eta_j \Big)\, d\eta \Big|^2 d\xi \tag{A.104}$$

$$+ \int_{\mathbb{R}^d_\xi} \Big| \lambda^1(\xi) \int_{\mathbb{R}^d_\eta} \widehat{u}(\xi-\eta)\, \widehat{\phi}(\eta) \Big(\sum_{j,k=1}^d \partial_{\xi_j}\partial_{\xi_k} m_\nu(\xi-\vartheta\eta)\, \eta_j\eta_k \Big)\, d\eta \Big|^2 d\xi. \tag{A.105}$$

Our assumption on the boundedness of the family $\{m_\nu\}_{\nu\geq 1}$ yields the existence of two constants C_{I}, C_{II} that do not depend on ν such that

$$\forall\, j \in \{1,\dots,d\} \qquad \text{it holds} \qquad \sup_{\nu\geq 1} \big|\partial_{\xi_j} m_\nu(\xi)\big| \leq C_{\mathrm{I}} \big(1+|\xi|\big)^{-1}, \tag{A.106a}$$

$$\forall\, j,k \in \{1,\dots,d\} \qquad \text{it holds} \qquad \sup_{\nu\geq 1} \big|\partial_{\xi_j}\partial_{\xi_k} m_\nu(\xi)\big| \leq C_{\mathrm{II}} \big(1+|\xi|\big)^{-2}. \tag{A.106b}$$

Moreover, due to Proposition A.3.3 we have that for all $j,k \in \{1,\dots,d\}$ it holds

$$\widehat{\phi}(\eta)\,\eta_j = \frac{1}{2\pi i}\widehat{\partial_j\phi}(\eta), \qquad \widehat{\phi}(\eta)\,\eta_j\eta_k = -\frac{1}{(2\pi)^2}\widehat{\partial_j\partial_k\phi}(\eta). \tag{A.107}$$

With this and the convolution estimate, we can easily estimate the term on the right hand side of (A.104) by

$$\text{(A.104)} \leq \int_{\mathbb{R}^d_\xi} \Big| \frac{C_{\mathrm{I}}}{2\pi i} \frac{\lambda^1(\xi)}{1+|\xi|} \int_{\mathbb{R}^d_\eta} \widehat{u}(\xi-\eta) \sum_{j=1}^d \widehat{\partial_j\phi}(\eta)\, d\eta \Big|^2 d\xi \leq \widetilde{C}_{\mathrm{I}} \|u\|_{L^2}^2 \sum_{j=1}^d \Big\|\widehat{\partial_j\phi}\Big\|_{L^1}^2. \tag{A.108}$$

Due to $\phi \in C_c^\infty(\mathbb{R}^d)$ we have $\widehat{\partial_j\phi} \in \mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$. Thus, (A.104) is bounded by $\|u\|_{L^2}^2$ times a constant depending on the supremum of the L^1-norms of $\widehat{\partial_j\phi}$ and C_{I}. Next, we analyze the term (A.105). Due to (A.106)–(A.107) we have

$$\text{(A.105)} \leq \int_{\mathbb{R}^d_\xi} \Big| \frac{C_{\mathrm{II}}}{(2\pi)^2} \lambda^1(\xi) \int_{\mathbb{R}^d_\eta} \widehat{u}(\xi-\eta)\, \big(1+|\xi-\vartheta\eta|\big)^{-2} \sum_{j,k=1}^d \widehat{\partial_j\partial_k\phi}(\eta)\, d\eta \Big|^2 d\xi. \tag{A.109}$$

Denoting with $g(\eta,\xi)$ the function

$$g(\eta,\xi) = \big(1+|\xi-\vartheta\eta|\big)^{-2} \sum_{j,k=1}^d \widehat{\partial_j\partial_k\phi}(\eta) \tag{A.110}$$

and applying Hölder's inequality on the η-integral, with $\widetilde{C}_{\mathrm{II}} := \frac{C_{\mathrm{II}}^2}{(2\pi)^4}$ we get the estimate

$$\text{(A.109)} \leq \widetilde{C}_{\mathrm{II}} \Big(\int_{\mathbb{R}^d_\xi} \big|\lambda^1(\xi)\big|^2 \int_{\mathbb{R}^d_\eta} |g(\eta,\xi)|^2\, d\eta\, d\xi \Big) \|\widehat{u}\|_{L^2(\mathbb{R}^d)}^2. \tag{A.111}$$

Next, we address the question, whether the integral

$$\int_{\mathbb{R}^d_\xi} \left|\lambda^1(\xi)\right|^2 \int_{\mathbb{R}^d_\eta} |g(\eta,\xi)|^2 \, d\eta \, d\xi \tag{A.112}$$

is finite. To this end, we make the following observations.

(i) Since the function ϕ is $C_c^\infty(\mathbb{R}^d)$ we have $\partial_j \partial_k \phi \in C_c^\infty(\mathbb{R}^d)$ for all $j,k \in \{1,\dots,d\}$. In particular, this implies that for all $j,k \in \{1,\dots,d\}$ we have $\widehat{\partial_j \partial_k \phi} \in \mathcal{S}(\mathbb{R}^d)$. Thus, for every $N \in \mathbb{N}$, there exists a constant C_N such that for all $j,k \in \{1,\dots,d\}$ we have $\widehat{\partial_j \partial_k \phi}(\eta) \leq C_N \big(1+|\eta|\big)^{-N}$.

(ii) Since the value $\vartheta \in (0,1)$ is fixed, we may estimate the term $\big(1+|\xi-\vartheta\eta|\big)^{-2}$ in the following way

$$\big(1+|\xi-\vartheta\eta|\big)^{-2} \leq \begin{cases} 4\big(1+|\xi|\big)^{-2} & \text{if } |\eta| \leq \frac{1}{2}|\xi| \\ 1 & \text{if } |\eta| \geq \frac{1}{2}|\xi|. \end{cases} \tag{A.113}$$

With the above considerations, we get the following estimate which holds for all $\xi \in \mathbb{R}^d_\xi$.

$$\begin{aligned} \int_{\mathbb{R}^d_\eta} |g(\eta,\xi)|^2 \, d\eta &= \int_{\{|\eta| \leq \frac{1}{2}|\xi|\}} |g(\eta,\xi)|^2 \, d\eta + \int_{\{|\eta| \geq \frac{1}{2}|\xi|\}} |g(\eta,\xi)|^2 \, d\eta \\ &\leq d^2 C_2 \int_{\{|\eta| \leq \frac{1}{2}|\xi|\}} \big(1+|\xi-\vartheta\eta|\big)^{-4} \big(1+|\eta|\big)^{-4} \, d\eta \\ &\quad + d^2 C_4 \int_{\{|\eta| \geq \frac{1}{2}|\xi|\}} \big(1+|\eta|\big)^{-8} \, d\eta \\ &\leq C\big(1+|\xi|\big)^{-4} \Bigg(\int_{\{|\eta| \leq \frac{1}{2}|\xi|\}} \frac{1}{\big(1+|\eta|\big)^4} \, d\eta + \int_{\{|\eta| \geq \frac{1}{2}|\xi|\}} \frac{1}{\big(1+|\eta|\big)^4} \, d\eta \Bigg) \\ &\leq \widetilde{C}\big(1+|\xi|\big)^{-4}. \end{aligned} \tag{A.114}$$

Here, the constant $\widetilde{C}$ depends on the dimension d, the constants C_2, C_4 and the value of the integral $\int_{\mathbb{R}^d} \big(1+|\eta|\big)^{-4} d\eta$. The number d^2 comes into play because this is the number of derivatives of second order. For the integral from (A.112) we thus get the estimate

$$\int_{\mathbb{R}^d_\xi} \left|\lambda^1(\xi)\right|^2 \int_{\mathbb{R}^d_\eta} |g(\eta,\xi)|^2 \, d\eta \, d\xi \leq \widetilde{C} \int_{\mathbb{R}^d_\xi} \big(1+|\xi|^2\big)\big(1+|\xi|\big)^{-4} \, d\xi. \tag{A.115}$$

Since the integral $\int_{\mathbb{R}^d_\xi} \frac{1+|\xi|^2}{(1+|\xi|)^4} \, d\xi$ is finite, (i.e. has the value C_{int}), we get from (A.111) the estimate

$$\widetilde{C}_{\text{II}} \Bigg(\int_{\mathbb{R}^d_\xi} \left|\lambda^1(\xi)\right|^2 \int_{\mathbb{R}^d_\eta} |g(\eta,\xi)|^2 \, d\eta \, d\xi \Bigg) \|\widehat{u}\|^2_{L^2(\mathbb{R}^d)} \leq \widetilde{C}_{\text{II}} \widetilde{C} \, C_{\text{int}} \, \|u\|^2_{L^2(\mathbb{R}^d)}. \tag{A.116}$$

Recalling estimate (A.108), we have thus shown that for arbitrary $u \in L^2(\mathbb{R}^d)$ there exists a constant $C \neq C(\nu)$ such that holds

$$\sup_{\nu \geq 1} \left\| T_\phi^\lambda u \right\|_{H^1(\mathbb{R}^d)}^2 \leq C \left\| u \right\|_{L^2(\mathbb{R}^d)}^2. \tag{A.117}$$

This was the claim. □

Remark A.3.16. *Theorem A.3.14 is also valid if the weakly convergent sequence $\{u_\nu\}_{\nu \geq 1}$ is N-dimensional with components $\{u_\nu^k\}_{k \in \{1,\dots,N\}}$ and if the family of pure multiplier symbols $\{m_\nu\}_{\nu \geq 1}$ is $N \times N$-dimensional with components $\{m_\nu^{jk}\}_{j,k \in \{1,\dots,N\}}$.*
In this case we have for all $j \in \{1, \dots, N\}$

$$\left(T_\phi^\nu u_\nu \right)_j = \sum_{k=1}^{N} \left[\mathsf{M}_\phi, m_\nu^{jk}(D) \right] u_\nu^k. \tag{A.118}$$

A.4. Helmholtz Decomposition

It is well known that every smooth, compactly supported function $u : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ can be decomposed into the sum of the gradient of a smooth scalar function φ and the curl of a smooth vector field A, i.e. $u = \nabla \varphi + \operatorname{curl} A$. Since for every smooth function $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ we have $\operatorname{curl} \nabla \varphi = 0$ and for every smooth vector field $A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ we have $\operatorname{div} \operatorname{curl} A = 0$, the function $u = \nabla \varphi + \operatorname{curl} A$ is decomposed into a curl-free (or irrotational) part $\nabla \varphi$ and a divergence-free (or solenoidal) part $\operatorname{curl} A$.

For vector fields $u \in L^2(\Omega; \mathbb{R}^d)$ with $\Omega \subseteq \mathbb{R}^d$ and $d \geq 2$ there exists a corresponding decomposition. We follow [Gal94, Sec. 1 of Ch. III] to give the precise result.

Definition A.4.1. *Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$. We define the space*

$$\mathcal{D}_\perp(\Omega) := \left\{ u \in C_c^\infty(\Omega; \mathbb{R}^d) : \operatorname{div} u = 0 \text{ in } \Omega \right\} \tag{A.119}$$

and denote by $L^2_\perp(\Omega)$ the completion of $\mathcal{D}_\perp(\Omega)$ w.r.t. the norm of $L^2(\Omega)$, i.e. we set

$$L^2_\perp(\Omega) := \overline{\mathcal{D}_\perp(\Omega)}^{\|\cdot\|_{L^2(\Omega)}}. \tag{A.120}$$

Moreover, we define the space

$$L^2_\parallel(\Omega) := \left\{ v \in L^2(\Omega; \mathbb{R}^d) : \exists\, w \in W^{1,2}_{\mathrm{loc}}(\Omega) \text{ with } v = \nabla w \right\}. \tag{A.121}$$

For the function spaces $L^2_\parallel(\Omega)$ and $L^2_\perp(\Omega)$, we have the following result.

Theorem A.4.2. *Let $d \geq 2$ and $\Omega \subseteq \mathbb{R}^d$ be an arbitrary domain. Then, we have*

(i) The spaces $L^2_{\parallel}(\Omega)$ and $L^2_{\perp}(\Omega)$ are orthogonal, i.e. for all $u \in L^2_{\perp}(\Omega)$ and for all $v \in L^2_{\parallel}(\Omega)$ it holds

$$\int_\Omega u \cdot v \, dx = 0. \tag{A.122}$$

(ii) For every $f \in L^2(\Omega;\mathbb{R}^d)$ there exist unique functions $u \in L^2_{\perp}(\Omega)$ and $v \in L^2_{\parallel}(\Omega)$ such that $f = u + v$. In particular, this means

$$L^2(\Omega) = L^2_{\parallel}(\Omega) \oplus L^2_{\perp}(\Omega). \tag{A.123}$$

We denote the projectors on the spaces $L^2_{\parallel}(\Omega)$ and $L^2_{\perp}(\Omega)$ by $\mathsf{P}_{\parallel}$ and $\mathsf{P}_{\perp}$, respectively.

(iii) In the case $\Omega = \mathbb{R}^3$, the projectors $\mathsf{P}_{\parallel}$ and $\mathsf{P}_{\perp}$ can be expressed as Fourier multiplier operators with symbols $\pi_{\parallel}$, $\pi_{\perp}$ and have the following representations in the Fourier domain

$$\pi_{\parallel}(\xi) := \frac{1}{|\xi|^2}\big(\xi \otimes \xi\big), \qquad \pi_{\perp}(\xi) := \frac{1}{|\xi|^2}\begin{pmatrix} \xi_2^2+\xi_3^2 & -\xi_1\xi_2 & -\xi_1\xi_3 \\ -\xi_1\xi_2 & \xi_1^2+\xi_3^2 & -\xi_2\xi_3 \\ -\xi_1\xi_3 & -\xi_2\xi_3 & \xi_1^2+\xi_2^2 \end{pmatrix}. \tag{A.124}$$

Proof. For (i), (ii) we refer to Section 1 of Chapter III in [Gal94], in particular [Gal94, Th. 1.1, p. 103]. In order to prove (iii) we introduce the matrix valued Fourier multipliers π_{div} and π_{curl} that correspond to the differential operators div and curl, by

$$\pi_{\mathrm{div}}(\xi) = 2\pi i \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \end{pmatrix}, \qquad \pi_{\mathrm{curl}}(\xi) = 2\pi i \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}. \tag{A.125}$$

Then, for every Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}^3;\mathbb{R}^3)$ we have

$$\begin{aligned}
\mathcal{F}\Big[\operatorname{curl} \mathcal{F}^{-1}\big[\pi_{\parallel}\,\mathcal{F}\varphi\big]\Big] &= \frac{2\pi i}{|\xi|^2}\begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}\begin{pmatrix} \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_2\xi_1 & \xi_2^2 & \xi_2\xi_3 \\ \xi_3\xi_1 & \xi_3\xi_2 & \xi_3^2 \end{pmatrix}\mathcal{F}\varphi \\
&= \frac{2\pi i}{|\xi|^2}\begin{pmatrix} -\xi_3\xi_1\xi_2+\xi_2\xi_1\xi_3 & -\xi_3\xi_2^2+\xi_2^2\xi_3 & -\xi_3^2\xi_2+\xi_2\xi_3^2 \\ \xi_3\xi_1^2-\xi_1^2\xi_3 & \xi_3\xi_1\xi_2-\xi_1\xi_2\xi_3 & \xi_1\xi_3^2-\xi_1\xi_3^2 \\ -\xi_2\xi_1^2+\xi_1^2\xi_2 & -\xi_2^2\xi_1+\xi_1\xi_2^2 & -\xi_2\xi_1\xi_3+\xi_1\xi_2\xi_3 \end{pmatrix}\mathcal{F}\varphi = 0 \\
\text{and}\quad \mathcal{F}\Big[\operatorname{div} \mathcal{F}^{-1}\big[\pi_{\perp}\,\mathcal{F}\varphi\big]\Big] &= \frac{2\pi i}{|\xi|^2}\begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \end{pmatrix}\begin{pmatrix} \xi_2^2+\xi_3^2 & -\xi_1\xi_2 & -\xi_1\xi_3 \\ -\xi_1\xi_2 & \xi_1^2+\xi_3^2 & -\xi_2\xi_3 \\ -\xi_1\xi_3 & -\xi_2\xi_3 & \xi_1^2+\xi_2^2 \end{pmatrix}\mathcal{F}\varphi \\
&= \frac{2\pi i}{|\xi|^2}\begin{pmatrix} \xi_1(\xi_2^2+\xi_3^2)-\xi_1\xi_2^2-\xi_1\xi_3^2 \\ -\xi_2\xi_1^2+\xi_2(\xi_1^2+\xi_3^2)-\xi_2\xi_3^2 \\ -\xi_3\xi_1^2-\xi_3\xi_2^2+\xi_3(\xi_1^2+\xi_2^2) \end{pmatrix}^T \mathcal{F}\varphi = \frac{2\pi i}{|\xi|^2}\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}\mathcal{F}\varphi = 0.
\end{aligned} \tag{A.126}$$

Thus, for every $\varphi \in \mathcal{S}(\mathbb{R}^3;\mathbb{R}^3)$ we have $\operatorname{curl}\big(\pi_\parallel(D)\varphi\big) = 0$ and $\operatorname{div}\big(\pi_\perp(D)\varphi\big) = 0$. Due to the density of $\mathcal{S}(\mathbb{R}^3;\mathbb{R}^3)$ in $L^2(\mathbb{R}^3;\mathbb{R}^3)$, this implies $\pi_\parallel(D)L^2(\mathbb{R}^3) \subset L^2_\parallel(\mathbb{R}^3)$ and $\pi_\perp(D)L^2(\mathbb{R}^3) \subset L^2_\perp(\mathbb{R}^3)$. Clearly, we have $\pi_\parallel + \pi_\perp = \mathsf{Id}_{3\times 3}$. This implies $\pi_\parallel(D)L^2(\mathbb{R}^3) = L^2_\parallel(\mathbb{R}^3)$ and $\pi_\perp(D)L^2(\mathbb{R}^3) = L^2_\perp(\mathbb{R}^3)$, i.e. $\pi_\parallel(D) = \mathsf{P}_\parallel$ and $\pi_\perp(D) = \mathsf{P}_\perp$. □

A.5. Symmetric Hyperbolic Systems

In this section we discuss symmetric hyperbolic systems with variable coefficients. For $N \in \mathbb{N}$ we denote the space of real symmetric $N\times N$-matrices with $\mathbb{R}^{N\times N}_{\text{symm}}$ and for brevity we denote the space $H^s(\mathbb{R}^d;\mathbb{R}^N)$ with $H^s(\mathbb{R}^d)$ in this chapter. Moreover, for $d \in \mathbb{N}$ and a Banach X we introduce

$$C_b^\infty([0,T]\times\mathbb{R}^d;X) := \Big\{\varphi \in C^\infty([0,T]\times\mathbb{R}^d;X) :\ \forall\, m \in \mathbb{N},\ \forall \alpha \in \mathbb{N}_0^{d+1}\ \exists\, K > 0 : \sum\nolimits_{|\alpha|\le m} \big\|\partial_x^\alpha \varphi\big\|_{C^0([0,T]\times\mathbb{R}^d;X)}\Big\}. \tag{A.127}$$

Definition A.5.1. *Let $T > 0$, $d \in \mathbb{N}$ and functions A_j, $B \in C_b^\infty([0,T]\times\mathbb{R}^d;\mathbb{R}^{N\times N}_{\text{symm}})$, $j \in \{1,\dots,d\}$ as well as an $\mathbb{R}^N$-valued function f depending on $(x,t) \in [0,T]\times\mathbb{R}^d$ be given. Then, we call the system of partial differential equations*

$$\partial_t u(x,t) + \sum_{j=1}^d A_j(x,t)\,\partial_{x_j} u(x,t) + B(x,t)\,u(x,t) = f(x,t) \tag{A.128}$$

a symmetric hyperbolic system. The number N is called the size of the system.

It seems that the first results on variable coefficient symmetric hyperbolic systems are due to K. O. Friedrichs, see [Fri54], [Fri58]. In the following we cite some simplified versions of results that can be found in the monograph [BeS07]. Namely D. Serre and S. Benzoni-Gavage deal with a situation where the matrix-valued functions A, B do not need to be symmetric, but are assumed to admit symmetrizers in certain senses. Other useful references on this topic are [Rau12], [Rac92] and [Tay96].

Theorem A.5.2 (existence and uniqueness, [BeS07, Th. 2.6])**.** *For arbitrary $s \in \mathbb{R}$, we take $T > 0$, $f \in L^2((0,T);H^s(\mathbb{R}^d))$ and $g \in H^s(\mathbb{R}^d)$. Moreover, let functions A_j, $B \in C_b^\infty([0,T]\times\mathbb{R}^d;\mathbb{R}^{N\times N}_{\text{symm}})$, $j \in \{1,\dots,d\}$ be given. Then, there exists a unique solution $u \in C^0([0,T];H^s(\mathbb{R}^d))$ to the Cauchy problem*

$$\partial_t u + \sum_{j=1}^d A_j\,\partial_{x_j} u + B\,u = f, \qquad u(0) = g. \tag{A.129}$$

Furthermore, there exists a constant $C > 0$ (depending on A_j, B, T, s) such that the following estimate holds

$$\forall\, t \in [0,T]: \qquad \big\|u(t)\big\|^2_{H^s(\mathbb{R}^d)} \le C\bigg(\big\|g\big\|^2_{H^s(\mathbb{R}^d)} + \int_0^t \big\|f(\tau)\big\|^2_{H^s(\mathbb{R}^d)}\,d\tau\bigg). \tag{A.130}$$

For $s < 1$, the notion of solutions is to be understood in the sense of distributions.

We define the differential operator L and its formally adjoint L^* by

$$\mathsf{L} := \partial_t + \sum_{j=1}^{d} A_j \partial_{x_j} + B, \qquad \mathsf{L}^* = -\partial_t - \sum_{j=1}^{d} \big(A_j \partial_{x_j} + \partial_{x_j} A_j\big) - B. \tag{A.131}$$

The function u is a solution to (A.129) in the sense of distributions iff for all $\psi \in C_c^\infty([0,T) \times \mathbb{R}^d; \mathbb{R}^N)$ it holds

$$\int_0^T \big\langle \mathsf{L}^* \varphi(t),\, u(t) \big\rangle_{H^{-s},H^s} \, dt = \int_0^T \big\langle \varphi(t),\, f(t) \big\rangle_{H^{-s},H^s} \, dt + \big\langle \varphi(0),\, g \big\rangle_{H^{-s},H^s}. \tag{A.132}$$

For $s \geq 1$, the solutions are to be understood in the strong sense.

In the case of constant coefficient matrices A_j, we have an energy balance for $s \geq 0$.

Proposition A.5.3. *For arbitrary $s \geq 0$ let functions $f \in L^2((0,T); H^s(\mathbb{R}^d))$ and $g \in H^s(\mathbb{R}^d)$ be given. Moreover, let $A_j \in \mathbb{R}^{N\times N}_{\text{symm}}$, $j \in \{1, \ldots, d\}$ be given*[17]*. Then, the unique solution $u \in C^0([0,T]; H^s(\mathbb{R}^d))$ to the Cauchy problem* (A.129) *admits the following energy balance*

$$\forall\, t \in [0,T] : \qquad \big\|u(t)\big\|^2_{L^2(\mathbb{R}^d)} = \big\|g\big\|^2_{L^2(\mathbb{R}^d)} + 2 \int_0^t \int_{\mathbb{R}^d} f(x,\tau) \cdot u(x,\tau) \, dx \, d\tau. \tag{A.133}$$

In order to prove this proposition, we need the estimate given in the following proposition. In fact this estimate is also a fundamental tool in the proof of Theorem A.5.2.

Proposition A.5.4 ([BeS07, Th. 2.1])**.** *Let functions A_j, $B \in C_b^\infty([0,T] \times \mathbb{R}^d; \mathbb{R}^{N\times N}_{\text{symm}})$, $j \in \{1, \ldots, d\}$ be given and let the differential operator L be defined as in* (A.131)*. Then, for all $s \in \mathbb{R}$ and $T > 0$ there exists $C > 0$ such that for $u \in C^0([0,T]; H^s(\mathbb{R}^d) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^d))$ we have the estimate*

$$\forall\, t \in [0,T] : \qquad \big\|u(t)\big\|^2_{H^s(\mathbb{R}^d)} \leq C \bigg(\big\|u(0)\big\|^2_{H^s(\mathbb{R}^d)} + \int_0^t \big\|\mathsf{L}u(\tau)\big\|^2_{H^s(\mathbb{R}^d)} \, d\tau \bigg). \tag{A.134}$$

Proof of Proposition A.5.3. We begin with the case $s = 1$. Due to

$$\partial_t u = -\sum_{j=1}^{d} A_j \, \partial_{x_j} u + f \tag{A.135}$$

we have $\partial_t u \in C^0([0,T]; L^2(\mathbb{R}^d))$. Thus, $u \in C^0([0,T]; H^1(\mathbb{R}^d)) \cap C^1([0,T]; L^2(\mathbb{R}^d))$. Multiplying equation (A.128) with u and integrating over $(0,t) \times \mathbb{R}^d$ for some arbitrary $t \in (0,T)$ yields

$$\frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \partial_t \big|u(x,\tau)\big|^2 + \sum_{j=1}^{d} A_j \, \partial_{x_j} \big|u(x,t)\big|^2 \, dx \, d\tau = \int_0^t \int_{\mathbb{R}^d} f(x,\tau) \cdot u(x,\tau) \, dx \, d\tau.$$

[17] For simplicity we assume $B \equiv 0$.

Due to the symmetry of the matrices A_j, the middle term vanishes and we may infer

$$\int_0^t \int_{\mathbb{R}^d} \partial_t \big|u(x,\tau)\big|^2 = 2\int_0^t \int_{\mathbb{R}^d} f(x,\tau)\cdot u(x,\tau)\,dx\,d\tau. \tag{A.136}$$

Performing the time integration on the left hand side yields (A.133). Since for $s \geq 1$ we have $H^s(\mathbb{R}^d) \subset H^1(\mathbb{R}^d)$, we have shown the claim for all $s \geq 1$.

For the case $s \in [0,1)$, we use mollifiers to construct approximating sequences $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{D}((0,T)\times\mathbb{R}^d;\mathbb{R}^N)$ and $\{g_k\}_{k\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^d;\mathbb{R}^N)$ satisfying the following strong convergences as $k \to \infty$

$$f_k \longrightarrow f \quad \text{in } L^2((0,T)\times\mathbb{R}^d;\mathbb{R}^N), \qquad g_k \longrightarrow g \quad \text{in } L^2(\mathbb{R}^d;\mathbb{R}^N). \tag{A.137}$$

Let u_k denote the corresponding solutions to (A.129) with data f_k and g_k. Due to the linearity of the Cauchy problem (A.129), the L^2-energy estimate (A.134) for $u-u_k$ shows the convergence

$$u_k \longrightarrow u \qquad \text{strongly in } C^0([0,T];L^2(\mathbb{R}^d)). \tag{A.138}$$

Moreover, for all $k \in \mathbb{N}$ we have

$$\forall\, t\in[0,T]: \qquad \big\|u_k(t)\big\|^2_{L^2(\mathbb{R}^d)} = \big\|g_k\big\|^2_{L^2(\mathbb{R}^d)} + 2\int_0^t\int_{\mathbb{R}^d} f_k(x,\tau)\cdot u_k(x,\tau)\,dx\,d\tau. \tag{A.139}$$

Taking the limit in (A.139) yields the assertion. □

Before we end this section with a typical example for a symmetric hyperbolic system, we introduce the "weak=strong" argumentation for the case $s = 0$. In fact, the same argument is used in the proof of Proposition A.5.2.

Lemma A.5.5. *Let $T > 0$, functions $g \in L^2(\mathbb{R}^d)$ and $f \in L^2((0,T);L^2(\mathbb{R}^d))$ as well as $A_j \in \mathbb{R}^{N\times N}_{\text{symm}}$, $j \in \{1,\dots,d\}$ be given. Assume that $u \in L^2((0,T);L^2(\mathbb{R}^d))$ is a distributional solution to the Cauchy problem*

$$\partial_t u + \sum_{j=1}^d A_j\,\partial_{x_j} u = f, \qquad u(0) = g, \tag{A.140}$$

i.e. for all test functions $\psi \in C_c^\infty([0,T)\times\mathbb{R}^d;\mathbb{R}^N)$ it holds

$$\int_0^T\int_{\mathbb{R}^d} u\cdot\partial_t\psi + \sum_{j=1}^d u\cdot A_j\partial_{x_j}\psi\,dx\,dt + \int_{\mathbb{R}^d} g(x)\cdot\psi(0,x)\,dx = -\int_0^T\int_{\mathbb{R}^d} f\cdot\psi\,dx\,dt. \tag{A.141}$$

Then, u is the unique solution from Theorem A.5.2 and it holds $u \in C^0([0,T];L^2(\mathbb{R}^d))$.

Proof. We proceed in three steps.

Step 1: The distribution $\mathcal{T}$ defined by

$$\mathcal{T}[\psi] := \int_0^T \int_{\mathbb{R}^d} \sum_{j=1}^d u \cdot A_j \, \partial_{x_j} \psi \, dx \, dt \tag{A.142}$$

is from the space $L^2((0,T); H^{-1}(\mathbb{R}^d))$. Therefore, if $u \in L^2((0,T); L^2(\mathbb{R}^d))$ satisfies (A.141), we have that the distribution $\partial_t u$ is also from the space $L^2((0,T); H^{-1}(\mathbb{R}^d))$ because for $\psi \in \mathcal{D}((0,T) \times \mathbb{R}^d; \mathbb{R}^N)$ the term with g is zero. In particular, this implies $u \in C^0([0,T]; H^{-1}(\mathbb{R}^d))$.
Step 2: We show that for $f \equiv 0$ and $g \equiv 0$ the function $u \equiv 0$ is the only possible solution to (A.140) in $C^0([0,T]; H^{-1}(\mathbb{R}^d))$. If $u \in C^0([0,T]; H^{-1}(\mathbb{R}^d))$ is a distributional solution to (A.140) with $f \equiv 0$ and $g \equiv 0$, we have that for all $\psi \in C_c^\infty([0,T) \times \mathbb{R}^d)$ it holds

$$\int_0^T \langle u, \partial_t \psi \rangle_{H^{-1},H^1} \, dt = -\sum_{j=1}^d \int_0^T \langle u, A_j \partial_{x_j} \psi \rangle_{H^{-1},H^1} \, dt. \tag{A.143}$$

Thus, the distribution $\partial_t u$ is from the space $C^0([0,T]; H^{-2}(\mathbb{R}^d))$. Due to $H^{-1}(\mathbb{R}^d) \subset H^{-2}(\mathbb{R}^d)$ it holds $u \in C^1([0,T]; H^{-2}(\mathbb{R}^d))$. Applying Proposition A.5.4 with $s = -1$ yields $u \equiv 0$. This implies that there exists exactly one solution $u \in C^0([0,T]; H^{-1}(\mathbb{R}^d))$ to the Cauchy problem (A.140).
Step 3: Due to $g \in L^2(\mathbb{R}^d)$ and $f \in L^2((0,T); L^2(\mathbb{R}^d))$, Theorem A.5.2 yields the existence of a solution $\widetilde{u} \in C^0([0,T]; L^2(\mathbb{R}^d))$. Since $C^0([0,T]; L^2(\mathbb{R}^d)) \subset C^0([0,T]; H^{-1}(\mathbb{R}^d))$ and uniqueness holds in the latter space, we may infer $u = \widetilde{u}$. This implies the regularity statement $u \in C^0([0,T]; L^2(\mathbb{R}^d))$. □

A typical example for a symmetric hyperbolic system of size 6 is the system consisting of the time dependent Maxwell equations in the case ϵ_0, $\mu_0 = 1$. Introducing the matrices

$$A_1 = \begin{pmatrix} & & & 0 & 0 & 0 \\ & 0_{3\times 3} & & 0 & 0 & 1 \\ & & & 0 & -1 & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & -1 & & 0_{3\times 3} & \\ 0 & 1 & 0 & & & \end{pmatrix}, \quad A_2 = \begin{pmatrix} & & & 0 & 0 & -1 \\ & 0_{3\times 3} & & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & & 0_{3\times 3} & \\ -1 & 0 & 0 & & & \end{pmatrix}$$

$$A_3 = \begin{pmatrix} & & & 0 & 1 & 0 \\ & 0_{3\times 3} & & -1 & 0 & 0 \\ & & & 0 & 0 & 0 \\ 0 & -1 & 0 & & & \\ 1 & 0 & 0 & & 0_{3\times 3} & \\ 0 & 0 & 0 & & & \end{pmatrix}, \quad B \equiv \begin{pmatrix} 0_{3\times 3} & 0_{3\times 3} \\ & \\ 0_{3\times 3} & 0_{3\times 3} \end{pmatrix}, \tag{A.144}$$

it is straightforward to see that by setting $u = \big(\mathbf{E}, \mathbf{H}\big)$, the Maxwell system given by

$$\partial_t \mathbf{E} = \operatorname{curl} \mathbf{H} \qquad \partial_t \mathbf{H} = -\operatorname{curl} \mathbf{E}. \tag{A.145}$$

can be written in the form (A.128) with $f = 0$.

A.6. Compensated Compactness

In this section we state two convergence results. The first result is due to Gérard (see [Gér91]) and the second result is a version of the classical div-curl lemma (see [CDM11]). Moreover, we show the embedding results needed in our preceding analysis for this context. All results are adapted to our notation and purpose.

We begin with a characterization of *microlocal defect measures*[18] (or H-measures).

Lemma A.6.1 (Characterization, [Gér91, Cor. 2.2]). *Let $\Omega \subseteq \mathbb{R}^N$ be open and let u, $\{u_k\}_{k\in\mathbb{N}} \subset L^2(\Omega;\mathbb{R}^N)$ be given such that*

$$u_k \longrightarrow u \qquad \textit{weakly in } L^2(\Omega;\mathbb{R}^N). \tag{A.146}$$

Let μ denote the microlocal defect measure[19] of the sequence $\{u_k\}_{k\in\mathbb{N}}$. Moreover, for some $m \in \mathbb{N}$ let $\mathsf{P} : L^2(\Omega;\mathbb{R}^N) \longrightarrow H^{-m}(\Omega;\mathbb{R}^N)$ be the differential operator defined by

$$\mathsf{P}u(x) = \sum_{|\alpha|\leq m} \partial_x^\alpha \Big(A_\alpha(x)\, u(x) \Big) \tag{A.147}$$

with coefficient functions $A_\alpha \in C^0(\Omega;\mathbb{R}^{N\times N})$ for a multi-index $\alpha \in \mathbb{N}_0^N$. We denote the principle part of the symbol of P with $p(x,\xi) = \sum_{|\alpha|=m} \xi^\alpha A_\alpha(x)$ and note that we have $p \in C^0(\Omega_x \times \mathbb{S}^{N-1}{}_\xi;\mathbb{R}^{N\times N})$. Then the following holds. If

$$\textit{the sequence } \{\mathsf{P}u_k\}_{k\in\mathbb{N}} \textit{ is relatively compact in } H^{-m}(\Omega), \tag{A.148}$$

then, the microlocal defect measure μ of the sequence $\{u_k\}_{k\in\mathbb{N}}$ and the principle part p of the symbol of P satisfy the relation[20]

$$\int_{\Omega_x} \int_{\mathbb{S}_\xi^{N-1}} p(x,\xi)\, \mu(x,\xi)\, d\xi\, dx = 0. \tag{A.149}$$

Microlocal defect measures have also been introduced by Luc Tartar in [Tar90]. He called them *H-measures*. Next, we state our first convergence result.

Theorem A.6.2 (Convergence theorem, [Gér91, Prop. 3.1]). *Let $\Omega \subseteq \mathbb{R}^N$ be open and let $\{u_k\}_{k\in\mathbb{N}}$ and $\{v_k\}_{k\in\mathbb{N}}$ be sequences in $L^2(\Omega;\mathbb{R}^N)$ such that*

$$u_k \longrightarrow u, \qquad v_k \longrightarrow v \qquad \textit{weakly in } L^2(\Omega;\mathbb{R}^N). \tag{A.150}$$

Let μ and ν denote their corresponding microlocal defect measures. If μ and ν are mutually singular, i.e. if disjoint Borel sets $A, B \subset \Omega_x \times \mathbb{S}_\xi^{N-1}$ exist with $\mu(A^c) = 0$, $\nu(B^c) = 0$, then, we have the convergence

$$\forall\, \varphi \in C_c^\infty(\Omega): \qquad \int_\Omega (u_k \cdot v_k)\, \varphi\, dx \longrightarrow \int_\Omega (u \cdot v)\, \varphi\, dx. \tag{A.151}$$

[18] For a definition, we refer to the original work [Gér91].

[19] The microlocal defect measure μ is a $\mathbb{C}^{N\times N}$-valued measure defined on $\Omega_x \times \mathbb{S}_\xi^{N-1}$.

[20] The integrand is a matrix multiplication, thus a $\mathbb{C}^{N\times N}$-valued matrix.

The second convergence result is the following version of the div-curl lemma given in the theorem below. In contrast to the classical version by Murat-Tartar, our assumptions on the div and curl of the corresponding sequences are somewhat weaker and we deal with weakly converging sequences in the reflexive Banach spaces L^p and L^q for p, $q \in (1,\infty)$ satisfying $p^{-1}+q^{-1}=1$. See [Mur78] for the original, though too restrictive version.

Theorem A.6.3 (Convergence theorem, [CDM11])**.** *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain and let $d \geq 2$. For $p \in (1,\infty)$ with conjugate exponent q, i.e. $p^{-1}+q^{-1}=1$. We consider two sequences $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^d)$ and $\{v_k\}_{k\in\mathbb{N}} \subset L^q(\Omega;\mathbb{R}^d)$ which satisfy*

$$u_k \longrightarrow u \quad \textit{weakly in } L^p(\Omega;\mathbb{R}^d), \qquad v_k \longrightarrow v \quad \textit{weakly in } L^q(\Omega;\mathbb{R}^d), \tag{A.152}$$

and

$$\operatorname{div} u_k \longrightarrow \operatorname{div} u \qquad \textit{strongly in } W^{-1,1}(\Omega), \tag{A.153a}$$

$$\operatorname{curl} v_k \longrightarrow \operatorname{curl} v \qquad \textit{strongly in } W^{-1,1}(\Omega;\mathbb{R}^{d\times d}). \tag{A.153b}$$

Furthermore, we assume that

$$\textit{the set} \quad \big\{u_k \cdot v_k\big\}_{k\in\mathbb{N}} \subset L^1(\Omega) \quad \textit{is equi-integrable.} \tag{A.154}$$

Then, we have the convergence

$$u_k \cdot v_k \longrightarrow u \cdot v \quad \textit{weakly in } L^1(\Omega). \tag{A.155}$$

The conditions $\operatorname{div} u_k \longrightarrow \operatorname{div} u$ strongly in $W^{-1,p}(\Omega)$ and $\operatorname{curl} v_k \longrightarrow \operatorname{curl} v$ strongly in $W^{-1,q}(\Omega;\mathbb{R}^{d\times d})$ are particularly satisfied for subsequences $\{\operatorname{div} u_{k_j}\}_{j\in\mathbb{N}}$ and $\{\operatorname{curl} v_{k_j}\}_{j\in\mathbb{N}}$ if these subsequences lie in compact subsets of the spaces $W^{-1,p}(\Omega)$ and $W^{-1,q}(\Omega;\mathbb{R}^{d\times d})$, respectively.

Lemma A.6.4. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Then, the following holds.*

(i) For all $p \in (1,\infty)$ and for all $d \in \mathbb{N}$ the embedding

$$L^p(\Omega) \hookrightarrow W^{-1,1}(\Omega) \qquad \textit{is compact.} \tag{A.156}$$

(ii) For all $m \in \mathbb{N}$, the embedding

$$H^{-m+1}(\Omega) \hookrightarrow H^{-m}(\Omega) \qquad \textit{is compact.} \tag{A.157}$$

Proof. For Banach spaces X, Y with duals X^*, Y^* a continuous linear operator $\mathsf{T} : X \longrightarrow Y$ is compact if and only if its adjoint operator $\mathsf{T}^* : Y^* \longrightarrow X^*$ is compact due to Schauder's theorem[21].

[21] See [Alt06, Th. 10.6, p. 387].

(i) We have $W^{-1,1}(\Omega) \cong \big(W_0^{1,\infty}(\Omega)\big)^*$.Therefore, it suffices to show the compactness of the embedding

$$W_0^{1,\infty}(\Omega) \hookrightarrow \big(L^p(\Omega)\big)^* \cong L^q(\Omega) \tag{A.158}$$

for q satisfying $p^{-1} + q^{-1} = 1$. Due to the Rellich-Kondrachov theorem[22], the above embedding is compact for all $q \in (1, \infty)$.

(ii) For all $m \in \mathbb{N}$ we have $H^{-m}(\Omega) \cong \big(H_0^m(\Omega)\big)^*$. Therefore, the compactness of the embedding[23] $H_0^{m-1}(\Omega) \hookrightarrow H_0^m(\Omega)$ yields the claim. □

A.7. Semigroup Theory

In this section we give a short collection of definitions and results from semigroup theory needed for our analysis.

Definition A.7.1 (dissipative operator, [Paz83, Ch. 1, Def. 4.1])**.** *Let X be a real Banach space and let X^* denote its dual. We define the non-empty duality set by*

$$F(x) := \big\{x^* \in X^* : \langle x^*, x\rangle_{X^*,X} = \|x\|^2 = \|x^*\|^2\big\}.$$

A linear operator $\mathsf{A} : X \supseteq D(\mathsf{A}) \longrightarrow X$ is called dissipative, if for every $x \in D(\mathsf{A})$ there exists an element $x^* \in F(x)$ such that $\langle x^*, \mathsf{A}x\rangle_{X^*,X} \leq 0$.

Theorem A.7.2 (corollary to the Lumer-Phillips theorem, [Paz83, Ch. 1, Cor. 4.4])**.** *Let X be a real Banach space, let X^* denote its dual and let $D(\mathsf{A}) \subseteq X$ be dense. Furthermore, let $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ be a closed linear operator and let $\mathsf{A}^* : D(\mathsf{A}^*) \longrightarrow X^*$ denote its adjoint. If both A and A^* are dissipative, then, A is the infinitesimal generator of a C_0-semigroup of contractions on X.*

Theorem A.7.3 (existence and uniqueness, [Bal77])**.** *Let X be a real Banach space, $T > 0$ and some function $f \in L^1((0,T); X)$ be given. For every $x \in X$ there exists a unique weak solution $u(t)$ of*

$$\partial_t u(t) = \mathsf{A}u(t) + f(t), \quad \textit{for } t \in (0, T] \tag{A.159}$$

in the sense of Definition A.7.4 satisfying $u(0) = x$ if and only if A is the generator of a C_0-semigroup $\{\mathsf{T}(t)\}_{t\geq 0}$ of bounded linear operators on X. In this case, $u(t)$ is given by

$$u(t) = \mathsf{T}(t)\, x + \int_0^t \mathsf{T}(t-s)\, f(s)\, ds, \quad \textit{for } t \in (0, T]. \tag{A.160}$$

[22] See [Dac08, Th. 12.12, p. 512].
[23] This version of the Rellich-Kondrachov theorem can for example be found in [Alt06, p. 328].

Definition A.7.4 (weak solutions, [Bal77, Definition])**.** *A function $u \in C^0([0,T];X)$ is called a weak solution of* (A.159) *if and only if for every $v \in D(\mathsf{A}^*)$ the function*

$$t \longmapsto \langle v, u(t)\rangle_{X^*,X}$$

is absolutely continuous on $[0,T]$ and for almost all $t \in [0,T]$ it holds

$$\frac{d}{dt}\langle v, u(t)\rangle_{X^*,X} = \langle \mathsf{A}^* v, u(t)\rangle_{X^*,X} + \langle v, f(t)\rangle_{X^*,X}.$$

Lemma A.7.5. *Let $\Omega \subseteq \mathbb{R}$ be an open set. We set $X = L^2(\Omega)$, $V = H^1(\Omega)$ as well as $\mathsf{A} := \frac{d}{dx}$. Then, the operator $\mathsf{A} : V \longrightarrow X$ is closed.*

Proof. We have to show that for $\{u_n\}_{n\in\mathbb{N}} \subset V$ with $u_n \longrightarrow u$ strongly in X, the convergence $\mathsf{A}u_n \longrightarrow y$ strongly in X for some $y \in X$ as well as $u \in V$ and $\mathsf{A}u = y$ hold.

Let $\{u_n\}_{n\in\mathbb{N}} \subset V$ and u, $y \in X$ be given and assume that the above convergences are satisfied. Then, for all $\varphi \in C_c^\infty(\Omega)$ we have

$$0 = \lim_{n\to\infty} \int_\Omega \big(\mathsf{A}u_n - y\big)\, \varphi \, dx = - \lim_{n\to\infty} \int_\Omega u_n(x)\, \varphi'(x) + y(x)\, \varphi(x)\, dx. \tag{A.161}$$

Thus, it holds

$$\int_\Omega u(x)\, \varphi'(x)\, dx = - \int_\Omega y(x)\, \varphi(x)\, dx. \tag{A.162}$$

By definition, this means $\mathsf{A}u = y$ and particularly implies $u \in V$. □

B. Notations

We fix $d \in \mathbb{N}$. The following notations are used throughout the thesis.

- The space $\mathbb{R}^3$ that corresponds to some quantity, gets the quantity as a subscript. For example $\mathbb{R}^3_{\mathbf{E}}$ is used for electric field vectors $\mathbf{E}$. On the other hand, the set $\mathbb{R}^3$ for the space variable x does not get a subscript.
- The set $\{x \in \mathbb{R}^d : \ |x| = 1\}$ is denoted with $\mathbb{S}^{d-1}$.
- For $N \in \mathbb{N}$, the set of Hermitian symmetric $N \times N$-matrices is given by

$$\mathbb{C}^{N\times N}_{\mathrm{herm}} := \left\{A = (a_{jk})_{j,k\in\{1,\dots,N\}} \in \mathbb{C}^{N\times N} : a_{jk} = \overline{a}_{kj}\right\}.$$

- We use the symbol $\forall_{\mathrm{a.a.}}$ to abbreviate *for almost all*.
- The symbol $\delta_{jk} := \begin{cases} 1, & \text{if } j = k \\ 0, & \text{else} \end{cases}$ is the Kronecker-δ.
- For a given set $\Omega \subseteq \mathbb{R}^d$, we write $(x,t) \in [0,T] \times \Omega$ and mean the pair (x,t) with $x \in \Omega$ and $t \in [0,T]$.
- For $m \in \mathbb{N}$ we denote by $\partial^m_{x_j}$ the partial differential operator $\frac{d^m}{dx_j^m}$.
- For a multi-index $\alpha = (\alpha_1, \dots \alpha_d) \in \mathbb{N}_0^d$ we denote by ∂^α_x the partial differential operator $\partial^{\alpha_1}_{x_1} \dots \partial^{\alpha_d}_{x_d}$.
- For a vector $(\mathrm{g}_1, \dots, \mathrm{g}_d) = \mathbf{g} \in \mathbb{R}^d$, the symbol $\nabla_{\mathbf{g}}$ denotes the following directional derivative $\nabla_{\mathbf{g}} := \sum_{j=1}^d \mathrm{g}_j \partial_{x_j}$.
- The Laplace operator Δ is defined by $\Delta := \sum_{j=1}^d \partial^2_{x_j}$.
- We denote the identity operator on any Banach space with Id. If the underlying Banach space is d-dimensional, we often write $\mathsf{Id}_{d\times d}$ instead.
- For $\Omega \subseteq \mathbb{R}^d$, a Banach space X and a Lipschitz continuous function $u : \Omega \longrightarrow X$, we denote with $\mathrm{Lip}(u,\Omega)$ the Lipschitz constant $\sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{\|u(x)-u(y)\|_X}{|x-y|}$.
- If a constant C depends on other constants c_1, c_2, ..., we write $C = C(c_1, c_2, \dots)$.
- For a Banach space X with dual X^* and $x \in X$ as well as $\xi \in X^*$ we denote the dual pairing with $\langle \xi, x\rangle_X$ or with $\langle \xi, x\rangle_{X^*,X}$.

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